Implementing Optimal Monetary Policy in New-Keynesian Models with Inertia

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Abstract

We consider optimal monetary policy in New Keynesian models with inertia. First order conditions, which we call the MJB-alternative, are found to improve upon the timeless perspective. The MJB-alternative is shown to be the best possible in the sense that it minimizes policymakers' *unconditional* expected loss, and further, it is numerically found to offer significant improvement over the timeless perspective. Implementation of the MJB-alternative is considered via construction of interest-rate rules that are consistent with its associated unique equilibrium. Following Evans and Honkapohja (2004), an expectations based rule is derived that always yields a determinate model and an E-stable equilibrium. Further, the "policy manifold" of all interest-rate rules consistent with the MJB-alternative is classified, and open regions of this manifold are shown to correspond to indeterminate models and unstable equilibria.

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1 Introduction

The benchmark, forward-looking, New-Keynesian models are closed by specifying the policy of the central bank. Preferences over different policy actions are typically modeled using a quadratic loss criterion, which is either taken as generic and so not tied to a particular specification of the economic model, or alternatively, may be derived as a second order approximation to average utility across private agents.[1]

Identification of the solution to the government's optimization problem depends on the surprisingly subtle definition of "optimality." The subtlety arises because, absent a commitment technology, the fully optimal rule is not time-consistent. To address this issue, Woodford (2003) advises the *timeless perspective*, which informally has policymakers ignore current (initial) conditions, and operationally imposes that policymakers choose inflation to satisfy a time-invariant first order condition (FOC). In effect, the policy maker implements the policy she would have chosen in the distant past.[2]

The timeless perspective advises setting policy in a time-invariant manner. On the other hand, the fully optimal solution to the government's problem sets inflation in the current period differently than in future periods: see Woodford (2003), Evans and Honkapohja (2006), and Section 3 below. As Jensen and McCallum (2002) point out, because of the policy maker's desire to behave differently in the current period, policy from the timeless perspective is not time-consistent, and in this sense, a commitment technology is implicitly assumed.

Given the assumption of a commitment technology capable of implementing the timeless perspective, one is naturally led to wonder whether there is a similar time-invariant policy that offers a superior performance, or even if there is a metric that makes this question meaningful. Noting that under certain conditions a reasonable policy implements an asymptotically stationary economy, Jensen and McCallum (2002), and separately, Blake (2002), suggest evaluating various policies by computing the expected value of the government's criterion across initial conditions, where the initial conditions are drawn from the associated asymptotic distribution. These authors have proposed a rule (the same rule, which we call the MJB-alternative) that, using this metric, improves upon Woodford's timeless perspective; and furthermore, Blake gives an argument showing the proposed rule is the best among rules of a given form.[3] Jensen and McCallum, using a calibrated ver-

[1] For a detailed analysis of the New-Keynesian model, its variants, and derivations of the associated welfare functions, see Woodford (2003).

[2] For a characterization of policy that is optimal from the timeless perspective, see Giannoni and Woodford (2002).

[3] Jensen and McCallum (2002) provide the rule only as an example of one superior to Woodford's, and do not explain how they obtained the rule.

sion of the model, show that the MJB-alternate is capable of implementing a 10.6% improvement over the timeless perspective.

The results of Jensen and McCallum (2002) and Blake (2002) are specific to the purely forward-looking New Keynesian model; however, substantial levels of inertia in both the AS and IS relations are often present in applied models of the economy. To assess the importance of the MJB-alternative's potential improvement over the timeless perspective, it is then necessary to compare their performances in models with inertia: this is the first task of our paper.

Blake's argument does not appear to extend to models with inertia. However, using alternative methods, we are able to generalize the MJB-alternative. We then analyze its performance for various levels of inertia and find that while under some calibrations of the model's parameters the improvement over the timeless perspective is insignificant, for other calibrations, and particularly when the government places priority on output stabilization, the improvement may be high – as high as 17% for calibrated models and much higher in case of strong serial correlation in the exogenous shocks. Interestingly, the presence of inertia in the model tends to mitigate this improvement, though the relationship between the degree of inertia and the level of mitigation is complex and non-monotonic.

The timeless perspective and the MJB-alternative are first order conditions which, together with the aggregate supply relation, define different "second best" rational expectations equilibria. As discussed in Section 3.1, some authors suggest taking these first order conditions as specific targeting rules, and thereby implicitly assume that the government's policy instrument is inflation. While this is reasonable for theoretical results and stylized models, practical implementation requires an alternate policy instrument. To operationalize policy implied by the FOCs, we impose that policymakers specify an interest-rate rule, often referred to as a T rule, which is consistent with the FOCs' implementation. The second task of our paper begins with the specification of interest-rate reaction functions designed to be consistent with optimal policy as determined by the timeless perspective or the MJB-alternative.

However, our second task does not end there. It is well-known that closing New-Keynesian monetary models with interest-rate rules may yield indeterminate steady states, and associated to these steady states are multiple sunspot equilibria: see Evans and McGough (2005a) and Evans and McGough (2005b). Since, in the presence of multiple equilibria, the equilibrium on which agents ultimately coordinate may exhibit unwanted properties, indeterminacy is undesirable. Therefore it is of considerable interest, when employing interest-rate reaction functions, to ascertain the determinacy properties of the model.

Even if the model has a unique REE under a given interest-rate rule, bad outcomes may obtain if boundedly rational agents are unable to coordinate on it. It is natural, then, for policymakers to desire that the equilibrium be stable under learning. It has been shown that for many models, including the models analyzed in this paper, stability under learning is governed by the notion of E-stability: for details, see Evans and Honkapohja (2001). Therefore it is necessary, when employing interest-rate reaction functions, to ascertain the E-stability properties of the model.

Having specified a collection of rules capable of implementing our FOC, we then examine their stability and determinacy properties. Similar examinations have been conducted in other models. In a purely forward-looking model, Evans and Honkapohja (2006) derive an interest-rate rule (which we call the "EH-rule") exhibiting dependence on agents' expectations, as well as on lagged endogenous variables and current shocks, which is consistent with the timeless perspective, and which always yields a stable and determinate steady state. We generalize this rule by deriving it for both the timeless perspective and the MJB-alternative when inertia is present in the model, and show that the resulting model is always determinate. While analytic results for E-stability are not available, we find numerically that for all calibrations and inertial specifications considered in our paper, as well as for both the timeless perspective and the MJB-alternative, the generalized EH-rule yields an E-stable equilibrium.

The generalized EH-rule is not the unique interest-rate reaction function consistent with the timeless perspective or the MJB-alternative. In fact, in the purely forward-looking model Evans and Honkapohja (2006) also derive what they called the "fundamentals based rule," which does not depend on agents' expectations, but is consistent with optimal policy from the timeless perspective. Further, they show that the resulting equilibrium is never E-stable and the associated steady state may be indeterminate. We complete the second task of the paper by extending this result and characterizing the manifold of all policy rules within a given class that are consistent with either the timeless perspective or the MJB-alternative; and numerically, we classify their stability and determinacy properties. We find that open regions of the manifold correspond to instability, indeterminacy, or both, as well as to the presence of sunspot equilibria that are stable under learning. This finding serves as a caution to policymakers interested in employing rules, like the generalized EH-rule, which are dependent on values of structural parameters: precise knowledge of these parameters may be required to avoid bad outcomes.

This paper is organized as follows: Section 2 reviews the model and theory needed to obtain and discuss the paper's main points, and in particular, the technical issues regarding determinacy and stability under learning are reviewed; Section 3 obtains, in the New Keynesian model with inertia, the first order conditions necessary to implement optimal policy from the timeless perspective, and presents the proposition describing the MJB-alternative; Section 4 focuses on implementable policy in the form of interest-rate reaction functions that are consistent with the timeless perspective or the MJB-alternative, and then examines the stability and determinacy properties of the New Keynesian model closed with these policy rules; and finally, Section 5 concludes. All proofs are contained in the Appendix.

2 Background

Here we discuss the necessary theory regarding determinacy and stability under learning, as well as present the hybrid New-Keynesian model of interest. These issues have been examined at length in other papers and so we keep our discussion brief. Because the model under investigation is quite similar, we borrow, in this review section, from Evans and McGough (2005b). A more detailed discussion of determinacy and learning as they pertain to these types of monetary models may be found in Evans and McGough (2005a).

2.1 Determinacy

The analysis of this paper is based on the New Keynesian model, which we present in Section 2.3. When closed with a specification of monetary policy – either an interest-rate rule or specific inflation targeting rule – the model yields a reduced form having the following expectational structure:

$$y_t = AE_t y_{t+1} + By_{t-1} + C\hat{g}_t. \qquad (1)$$

Here y_t is a vector of endogenous variables and $\hat{g}_t$ is exogenous fundamental noise, which we assume to be a stationary VAR(1) with damping matrix ρ.

A rational expectations equilibrium (REE) is any non-explosive process $\overline{y}_t$ satisfying (1)[4].

Existence and uniqueness of REE depend on the model's parameters. As is standard, we say that the model is determinate if there is a unique REE, indeterminate if there are many REE, and explosive otherwise. Methods for assessing whether a model is determinate, indeterminate, or explosive are well known and we refrain from discussing them here: see Evans and McGough (2005a) for a detailed analysis of determinacy in models of the form (1).

In case the model is determinate, the unique equilibrium has the form

$$y_t = \bar{b} y_{t-1} + \bar{c} \hat{g}_t, \tag{2}$$

where the matrices $\bar{b}$ and $\bar{c}$ may be computed using, for example, the method of undetermined coefficients. If the model is indeterminate there will still be at least one REE of the form (2), and we will call them minimal state variable (MSV) solutions.[5] However, there will also be other REE that exhibit dependence on extrinsic stochastic processes: these processes may be thought of as capturing rational forecast errors, and are sometimes called sunspots. For a detailed discussion of sunspot equilibria in models of the form (1) see Evans and McGough (2005a).

2.2 Learning

If the model is determinate, so that there is a unique REE, it is desirable that the solution be stable under learning. By stability we mean that if agents estimate the parameters of a forecasting model using least squares regression and form their expectations accordingly, then the economy will eventually converge to the equilibrium. Because the models are self-referential, that is, because the evolution of the economy depends on how agents form expectations, the stability of an REE under least squares learning cannot be taken for granted.

[4] The notion of a non-explosive process can be made precise with several alternate definitions: see Evans and McGough (2005c) for details.

[5] If the model is indeterminate there may be multiple equilibria of the form (2), depending on whether certain eigenvalues are real or complex. Bennett McCallum, who introduced the term "MSV-solution" in McCallum (1983), proposes an additional selection criterion so that the MSV-solution, by his definition, is always unique.

To fix ideas, assume the model is determinate and that agents have a forecasting model of the form

$$y_t = a + by_{t-1} + cg_t \tag{3}$$

The functional specification of the forecasting model is often referred to as a Perceived Law of Motion (PLM), and is typically taken to have a form consistent with the REE of interest.[6] Under learning agents obtain least squares estimates (a_t, b_t, c_t) of the PLM's parameters using data through time t, and then use the estimated PLM to form their forecasts $E_t^* y_{t+1}$, which in turn influences the path of y_t through the reduced form model (1).[7] The question is whether $(a_t, b_t, c_t) \to (\bar{a}, \bar{b}, \bar{c})$ as $t \to \infty$. If so, we say that the solution is stable under learning.

Analysis of stability under learning is usually done using expectational stability (E-stability). This is because, for a wide range of models and solutions, E-stability has been shown to govern the local stability of REE under least squares learning. In many cases this correspondence can be proved, and in cases where it cannot be formally demonstrated the "E-stability principle" has been validated through simulations. For a thorough discussion of E-stability see Evans and Honkapohja (2001).

The E-stability technique is based on a mapping from the PLM to the corresponding Actual Law of Motion (ALM) parameters. For the case at hand, if agents believe in the PLM identified by (a, b, c) then their corresponding forecasts are given by $E_t^* y_{t+1} = a + bE_t^* y_t + cE_t^* \hat{g}_{t+1}$. Using $E_t^* y_t = a + by_{t-1} + c\hat{g}_t$,

[6] The functional form of (3) clearly nests that of the REE (2), but also includes a constant term a. The REE lacks a dependence on a constant only because it has been written in deviation from steady-state form. By including a constant term in agents' PLM, we are imposing the natural assumption that they do not know the value of the steady-state, but rather, they must learn it, which is equivalent to learning that the REE value of the perceived parameter a is zero. It is well known that the stability of a particular REE may hinge on the presence of a constant in the PLM, especially when the model is indeterminate.

[7] The reduced form model (1), when capturing policy determined by an interest-rate rule, incorporates private agents' consumption Euler equation, thus this type of stability analysis is sometimes called *Euler equation learning*. The assumption is that agents form expectations of economic aggregates and then behave in a way that is consistent with (a linearization of) their Euler equation. Alternate methods of stability analysis have been suggested; most notably, Preston (2003) suggests that agents should be modeled as behaving in a way consistent with their lifetime budget constraint. Whether a given REE is stable under learning may depend on the way in which the stability analysis is conducted; for simplicity, we focus on Euler equation learning for this study.

and assuming for convenience that ρ is known so that $E_t^* \hat{g}_{t+1} = \rho \hat{g}_t$, yields

$$E_t^* y_{t+1} = (I_2 + b)a + b^2 y_{t-1} + (bc + c\rho)\hat{g}_t.$$

Inserting $E_t^* y_{t+1}$ into (1) and solving for y_t as a linear function of an intercept, y_{t-1} and $\hat{g}_t$ yields the corresponding ALM:

$$y_t = A(I_2 + b)a + (Ab^2 + B)y_{t-1} + (A(bc + c\rho) + C)\hat{g}_t,$$

which describes the actual evolution of y given the beliefs (a, b, c). The map from perception to reality, then, is given by

$$a \rightarrow A(I_2 + b)a \quad (4)$$
$$b \rightarrow Ab^2 + B \quad (5)$$
$$c \rightarrow A(bc + c\rho) + C. \quad (6)$$

To simplify notation, combine the regressors into the vector

$$X_t' = (1, y_{t-1}', \hat{g}_t')$$

and write the perceived parameters as $\theta = (a, b, c)'$. Then the PLM can be written as $y_t = \Theta' X_t$, and equations (4)-(6) define a mapping from PLM parameters Θ to the ALM parameter $T(\Theta)$. The REE $\bar{\Theta} = (\bar{a}, \bar{b}, \bar{c})$ is a fixed point of this map and it is said to be E-stable if it is locally asymptotically stable under the differential equation

$$\frac{d\Theta}{d\tau} = T(\Theta) - \Theta. \quad (7)$$

The E-stability principle tells us that E-stable REE are locally learnable for Least Squares and closely related algorithms. That is, if Θ_t is the time t estimate of the coefficient vector Θ, and if Θ_t is updated over time using recursive least squares, then $\bar{\Theta}$ is a possible convergence point, i.e. locally $\Theta_t \rightarrow \bar{\Theta}$, if and only if $\bar{\Theta}$ is E-stable. Computing E-stability conditions is often straightforward, involving computation of eigenvalues of the Jacobian matrices of (7).

The discussion so far has been restricted to the determinate case; however, it is (relatively) straightforward to extend these ideas to sunspot equilibria in case of model indeterminacy. Also, in the indeterminate case, stability under learning of a given equilibrium may depend on the functional form of the forecasting model used by agents when forming their expectations. Because these issues are not central to our work here, we refrain from their discussion and simply refer readers to Evans and McGough (2005a).

2.3 The New Keynesian Model

We study a hybrid version of the New Keynesian monetary model as given by

$$IS: x_t = -\phi(i_t - E_t\pi_{t+1}) + \delta E_t x_{t+1} + (1-\delta)x_{t-1} + g_t \quad (8)$$
$$AS: \pi_t = \beta(\gamma E_t\pi_{t+1} + (1-\gamma)\pi_{t-1}) + \lambda x_t + u_t. \quad (9)$$

Here x_t is the proportional output gap, π_t is the inflation rate, and g_t and u_t are independent, exogenous, stationary, zero mean AR(1) shocks with damping parameters $0 \leq \rho_g < 1$ and $0 \leq \rho_u < 1$ respectively.

The first equation is a formulation of the forward-looking IS curve amended to include inertia. This functional form may be obtained from a linearized model of optimization behavior on the part of consumers. In some cases we also allow for an inertial term x_{t-1}, which is present due to habit formation: see for example Smets (2003). The second equation is the forward-looking Phillips curve. When $\gamma = 1$, equation (9) is the pure forward-looking New Keynesian "AS" relationship based on "Calvo pricing," and employed in Clarida, Gali, and Gertler (1999) and Ch. 3 of Woodford (2003).[8] Here $0 < \beta < 1$ is the discount factor. Again, this equation is obtained as the linearization around a steady state. The specification of the AS curve in the case $0 < \gamma < 1$ incorporates an inertial term and is similar in spirit to Fuhrer and Moore (1995), the Section 4 model of Gali and Gertler (1999), and the Ch. 3, Section 3.2 model of Woodford (2003), each of which allows for some backward looking elements. Models with $0 < \gamma < 1$ are often called "hybrid" models, and we remark that in some versions, such as Fuhrer and Moore (1995), $\beta = 1$, so that the forward and backward looking components sum to one, while in other versions $\beta < 1$ is possible.

The model may be closed by specifying monetary policy, and we consider alternate optimal specifications below. Once a policy has been specified, the model may be placed in the reduced form (1), and its determinacy and stability properties analyzed. While some analytic results are available, numeric methods must be used in the general case, and this requires assigning values to the model's parameters. We consider four calibrations of the parameters in the IS-AS curves, as due to Woodford (1999), Clarida, Gali, and Gertler (2000), McCallum and Nelson (1999), and McCallum and Jensen (2002): the relevant parameter values are given in Table 1.

[8] For the version with mark-up shocks see Woodford (2003) Chapter 6, Section 4.6.

Table 1: Calibrations

Name	ϕ	λ
W	1/.157	.024
CGG	4	.075
MN	.164	.3
MJ	.164	.02

Table 2: Inertial Specifications

Name	γ	δ
Forward	1	1
Small Lag	.75	.75
Medium Lag	.5	.5
Large Lag	.25	.25
Full Lag	.01	.01

With these calibrations, we consider five inertial specifications, as given by Table 2. Finally, for each calibration and inertial specification, we will consider discount values of $\beta = .98$, $\beta = .99$ and $\beta = 1$.

3 Optimal Policy

We model the preferences of policymakers over alternate equilibrium paths as given by

$$E_t \left(\sum_{k=0}^{\infty} \beta^k (\psi x_{t+k}^2 + \pi_{t+k}^2) \right). \tag{10}$$

For some specifications of the New Keynesian model, a loss criterion of this form may be taken as capturing a second order approximation to aggregate private agent utility, in which case ψ is a function of the model's deep parameters.[9] Rather than fixing the link between government preferences and the

[9]The loss criterion associated to models with inertia may include a dependence on lagged endogenous variables.

underlying model, we take (10) as a generic loss function parameterized by ψ, which simply captures the relative importance of output stabilization over inflation control. Note we have assumed, for simplicity, that the government and private agents have the same discount factor.

To construct optimal monetary policy, we begin by identifying the optimal REE, that is the REE which minimizes the government's loss subject to the Phillips curve constraint. In this sense, we are implicitly assuming π_t to be the government's instrument: see Section 3.1 for more discussion. This assumption makes the IS relation superfluous, and the associated reduced form model is obtained by combining aggregate supply with a first order restriction characterizing government behavior. The associated time-path of output and inflation represents the optimal REE consistent with the government's objective and the AS-relation. In Section 4, we will use the IS relation to establish the time-path of the interest-rate consistent with this optimal REE. Interpreting this time-path as an interest-rate rule will then operationalize our optimal policy.

Taking π to be the instrument, the government's problem is to minimize (10) subject to (9) and the initial conditions π_{t-1} and x_{t-1} (and u_{t-1}). Using Lagrange's method (and discounting the constraint appropriately), we obtain the following first order conditions (see Appendix for details):

$$\omega_{t+k} = \frac{2\psi}{\lambda} x_{t+k} \qquad (k \geq 0) \qquad (11)$$

$$2\pi_{t+k} = -\omega_{t+k} + \gamma\omega_{t+k-1} + \beta^2(1-\gamma)E_{t+k}\omega_{t+k+1} \qquad (k > 0) \qquad (12)$$

$$2\pi_t = -\omega_t + \beta^2(1-\gamma)E_t\omega_{t+1}, \qquad (13)$$

where ω_t represents the Lagrange multiplier at time t. Equations (11) - (13) characterize the fully optimal solution to the government's problem. That this solution is not time consistent is evident from equations (12) and (13), which instruct the policy-maker to behave differently in the initial period than in subsequent periods.

3.1 The Timeless Perspective

To address the inconsistency problem, Woodford advises the timeless perspective, which has policy-makers ignore (13). Equations (11) and (12) may then be combined to obtain the following relation between current inflation and current, lagged and expected future output gap:

$$\pi_t = -\frac{\psi}{\lambda}\left(x_t - \gamma x_{t-1} - \beta^2(1-\gamma)E_t x_{t+1}\right). \tag{14}$$

We note that in case $\gamma = 1$, (14) provides the same first order condition as obtained by Evans and Honkapohja (2006) in case of no inflation inertia.

Policy described by (14) is time consistent in the sense that it is time-invariant (unlike the policy described by (11) - (13)) and optimal in the sense that it is the policy an optimizing policy maker from the distant past would have chosen for today. However, it is not time consistent in the sense of Kydland and Prescott (1977): each period policymakers have an incentive to deviate from the planned policy.[10] For more discussion on the timeless perspective, see Woodford (2003), McCallum (2005), and Dennis (2001).

Until now we have assumed, somewhat unrealistically, that inflation is the government's policy instrument, and that it can be set to satisfy (14). Indeed, some authors suggest taking the first order condition (14) as a specific targeting rule. Interpreted this way, and assuming policymakers can, in some way not explicitly modeled, impose that (14) holds at every point in time, then (9) and (14) comprise a fully specified reduced form model. Furthermore, arguments in Woodford (2003) guarantee this model to be determinate, and hence yield a unique rational expectations equilibrium.[11] Whether the unique REE is stable under learning must be determined numerically. We find that for all calibrations and inertial specifications, and with all discount factors, expectational stability always obtains.

Because the specific targeting rule interpretation does not provide a model of policy maker behavior, some ambiguity remains concerning precisely how monetary policy is implemented. We address this issue in Section 4. There we design interest-rate reaction functions consistent with the optimal REE defined by the timeless perspective or the MJB-alternative. However, like Evans and Honkapohja (2006), we find that such policy rules do not necessarily imply (14) is satisfied, and so the issues of determinacy and stability

[10]If the model is determinate and the equilibrium is E-stable, then there is an additional sense in which the timeless perspective is time consistent when agents form their expectations based on least-squares learning: provided policymakers resist following discretionary policy, they do not have to worry about private agents "rationally" anticipating discretionary behavior: see McCallum (1995).

[11]See Footnote 8 on page 542.

must be revisited.[12] Because of this, we recommend choosing instrument rules consistent with (14) that also result in a determinate model with an E-stable equilibrium.

3.2 The MJB-Alternative

As noted above, the timeless perspective implicitly assumes a commitment technology. Furthermore, while it is optimal from the perspective of a policy maker in the distant past, one is led to wonder how good the timeless perspective is from a current view point, or even if there is a natural metric that may be used to address this question. McCallum and Jensen (2002), and separately, Blake (2002), suggest considering the average value of the government's objective, where the average is taken across initial conditions drawn with respect to their asymptotic distribution. These authors then examine, in the purely forward-looking case (i.e. $\gamma = 1$), whether there are rules exhibiting the same linear dependence as (14), (i.e. incorporating the same commitment technology) but which yield lower average losses. Both Jensen and McCallum and Blake show numerically that the relationship

$$\pi_t = -\frac{\psi}{\lambda}(x_t - \beta x_{t-1}) \tag{15}$$

is superior to the timeless perspective: Jensen and McCallum simply state it as an example of an improvement; but Blake proceeds to show that in fact (15) is the optimal rule of that form. As we noted in the introduction, the method Blake uses to show his rule is optimal is not tractable in case $\gamma < 1$. However, in the Appendix, we present an alternate method that allows us to derive an FOC of the form (14) minimizing the average government loss. In this subsection we make precise the notion of optimality, state the optimal rule, and then analyze its implications for determinacy and stability, and in the next subsection we compare its outcomes to those of the timeless perspective.

Consider the following linear relationship between inflation and output gap:

$$\pi_t = \theta_0 x_{t-1} + \theta_1 x_t + \theta_2 E_t x_{t+1}. \tag{16}$$

Equations (9) and (16) comprise a fully specified reduced form model. Let Θ be the set of all $\theta \in \mathbb{R}^3$ for which this reduced form model is determinate.

[12]Policy rules consistent with the timeless perspective's first order condition may impart indeterminacy and some of the associated equilibria will not satisfy (14).

We know Θ is not empty because the timeless perspective defines one of its elements. Given $\theta \in \Theta$, we may compute the associated unique REE

$$y_t = b(\theta)y_{t-1} + c(\theta)u_t. \tag{17}$$

This process is stationary, and therefore we may compute its second moments. Provided the initial conditions are assumed taken from the relevant asymptotic distribution, choosing policy θ to minimize the average value of the government's loss (10) is the same as choosing θ to solve

$$\min_{\theta \in \Theta} \psi E(x^2|\theta) + E(\pi^2|\theta). \tag{18}$$

Using (17), these second moments can be computed in terms of θ, and hence the problem solved. This is the method used by Blake in the forward-looking case. Unfortunately, computing $b(\theta)$ and $c(\theta)$ explicitly in case $\gamma < 1$ does not appear tractable, and therefore the relevant first order conditions can not be written down. In the Appendix, we offer an alternate method of obtaining the optimal FOC among rules of the form (16), by proving the following result.

Proposition 1 *There exists $\bar{\beta} \in (0,1)$ so that for $\beta \in (\bar{\beta}, 1]$ the reduced form model given by (9) and*

$$\pi_t = -\frac{\psi}{\lambda}\left(x_t - \beta\gamma x_{t-1} - \beta(1-\gamma)E_t x_{t+1}\right) \tag{19}$$

is determinate and its unique stationary REE solves the minimization problem (18).

Analytic results for E-stability of the unique REE given by (9) and (19) are not available, but we find numerically that for all calibrations and inertial specifications, the associated REE is stable under learning.[13]

Notice that in case $\gamma = 1$ so that there is no inertia in the Phillips curve, (19) reduces to (15), the condition suggested by McCallum and Jensen, and established as optimal by Blake. Notice, too, that for $\beta = 1$, (19) and the timeless perspective are identical as expected. We call (19) the "MJB-alternative."

[13]Here were are interpreting (19) as a specific targeting rule, and again, concerns regarding its implementation apply. We address these concerns by constructing explicit interest-rate rules in Section 4.

3.3 Comparing the Timeless Perspective and the MJB-Alternative

The results of the previous section indicate that the MJB-alternative (19) at least weakly dominates the timeless perspective (14); however, the MJB-alternative is simply a small continuous transformation of the timeless perspective, and so it is quite natural to wonder how much improvement is obtained. This question has been considered by Jensen and McCallum (2002) in the non-inertial case, with $\lambda = .02$, for varying values of ψ, β, and ρ. They obtain a maximum improvement of 10.26%. We conducted a similar analysis, and some of the results we obtained are reported in Table 3.

As expected, the MJB-alternative always yields an improvement over the timeless perspective, and in the forward-looking case, with heavy weight place on output stabilization, the improvement may be quite significant. Interestingly, the presence of inertia seems to mitigate the second-best nature of the timeless perspective: for a hybrid model with $\gamma = .5$, the maximum improvement across calibrations is only .06%. Further examination of this mitigating effect indicates a complex, non-monotonic pattern. Consider Figure 1, which plots the percent improvement of the MJB-alternative over the timeless perspective for the Woodford calibration and varying γ. As γ increases to .5, which corresponds to the Medium Lag inertial specification, the improvement drops from 1.5% to near zero before rising again. This pattern is qualitatively the same as seen across calibrations, with the minimum improvement obtaining near $\gamma = .5$.

Fig 1 here

The magnitude of the serial correlation in the markup shock also impacts the improvement of the MJB alternative over the timeless perspective, with the numerically obtained qualitative result being that for most calibrations and inertia specifications, larger serial correlation leads to larger improvement; however, for inertial specifications with γ near .5, the relationship between serial correlation and improvement is again non-monotonic: see Figure 2 as an example.

Fig 2 here

For high serial correlation, the improvement may be very large: for example, if the MJ calibration is used under the forward inertial specification, then

Table 3: TP/MJB Comparison (Forward Model and Medium Lag, $\beta = .98$)

	Forward Model				Medium Lag			
ψ	W	CGG	MN	MJ	W	CGG	MN	MJ
.1	2.42%	.45%	.04%	3.08%	.04%	.01%	.00%	.04%
1	8.76%	2.46%	.30%	10.16%	.06%	.04%	.01%	.05%
10	16.12%	8.81%	1.79%	17.08%	.01%	.05%	.03%	.01%

with $\beta = .99$, $\psi = 10$ and $\rho = .999$ the improvement is 1882%. We conclude that the a-priori minor modification to the timeless perspective advocated by the MJB-alternative can yield significantly superior results in terms of average government loss, and so must be taken seriously.

4 Implementing Optimal Policy

We have characterized the optimal REE via first order conditions, taking the AS relation as a constraint. Our goal now is to investigate how the optimal REE can be implemented. That is, we look for interest-rate reaction functions that, when combined with the AS and IS relations, result in the optimal REE attaining according to determinacy and E-stability. To this end, we "nest" the timeless perspective and the MJB-alternative in the following generalized FOC:

$$\pi_t = -\frac{\psi}{\lambda}\left(x_t - \xi\gamma x_{t-1} - \frac{\beta^2}{\xi}(1-\gamma)E_t x_{t+1}\right), \tag{20}$$

where $\xi = 1$ in case of the timeless perspective and $\xi = \beta$ in case of the MJB-alternative.

4.1 The Fundamentals Rule

As noted above, for β near 1, the system of expectational difference equations (9) and (20) is determinate for either value of ξ. Letting $y = (x, \pi)'$, the unique stationary equilibrium can be written

$$y_t = by_{t-1} + cu_t. \tag{21}$$

Now recall the IS-relation (8), repeated here for convenience:

$$x_t = -\phi(i_t - E_t\pi_{t+1}) + \delta E_t x_{t+1} + (1-\delta)x_{t-1} + g_t. \tag{8}$$

To create an interest-rate rule consistent with the optimal REE, we follow Evans and Honkapohja (2006) (EH) and assume agents form expectations using (21). Imposing these expectations into the IS relation (8), we may then solve for i_t, thus obtaining an interest-rate rule of the form

$$i_t = \tilde{\alpha}^L y_{t-1} + \tilde{\alpha}^{\hat{g}} \hat{g}_t. \tag{22}$$

We call (22) the "fundamentals rule." The values of the 1×2 matrices $\tilde{\alpha}^L$ and $\tilde{\alpha}^{\hat{g}}$ depend on the model's reduced form parameters as well as which FOC is used.

We may now consider the full reduced form model, assuming monetary policy is implemented by following the fundamentals rule. The model is given by (9), (8), and (22). Results concerning a special case of this model are known. Evans and Honkapohja (2006) obtained the fundamentals rule in case $\gamma = 1$ and the timeless perspective is assumed. They showed analytically that the model may be indeterminate, and the equilibria are always unstable under learning. Similarly, we find

Proposition 2 *In case $\gamma = \delta = 1$ and the MJB-alternative is used, under the fundamentals based rule, the economy may be indeterminate, and the equilibria are always unstable under learning.*

The proof of this proposition mimics the proof by EH, and we suppress the details.

Similar work can be done in case of inertia; however, we must proceed numerically as analytic results are unavailable. Some of our results are collected in Tables 4 and 5. Here $\beta = .99$ and the MJB-alternative is used. To identify the stability and determinacy properties, we use the notation SD (stable determinacy), UD (unstable determinacy), SI (stable indeterminacy, that is, E-stable sunspot equilibria), and UI (unstable indeterminacy). Similar results obtain for other calibrations and inertial specifications and for the timeless perspective.

Interpreting the first order conditions given by the timeless perspective or the MJB-alternative as specific targeting rules, and assuming that policy-makers are in some unspecified way able to achieve the specific targeting rule

Table 4: Fundamentals Rule (Small Lag and Medium Lag)

	Small Lag				Medium Lag			
ψ	W	CGG	MN	MJ	W	CGG	MN	MJ
.1	UI	UI	SD	UI	UI	UI	SD	SD
1	UD	UD	UI	UI	UI	UI	SD	UI
10	UD	UD	UI	UI	UI	UD	UI	UI

Table 5: Fundamentals Rule (Large Lag and Full lag)

	Large Lag				Full Lag			
ψ	W	CGG	MN	MJ	W	CGG	MN	MJ
.1	SD	UI	SD	SD	SD	SD	SD	SD
1	UI	UI	SD	SD	SD	SD	SD	SD
10	UI	UI	SD	SD	SD	SD	SD	SD

so that only the AS relation is considered as a restriction, both the timeless perspective and the MJB-alternative resulted in stable determinacy. The work of Evans and Honkapohja (2006) and the results here show that simply using the IS relation to design an interest-rate rule consistent with the associated optimality condition can be destabilizing and hence ill-advised.

4.2 The EH-rule

The results reported in Proposition 2 and Tables 4 and 5 warn against the use of the fundamentals rule. Evans and Honkapohja, in case $\gamma = 1$ and under the timeless perspective, faced a similar problem, and proposed the following solution. Instead of forming a rule by first computing expectations, EH suggest taking expectations as given. Specifically, combine the general FOC (20) and the Phillips curve (9) and solve for x_t to obtain

$$x_t = \frac{\lambda}{\psi + \lambda^2}\left(\frac{\psi\xi\gamma}{\lambda}x_{t-1} + \frac{\psi\beta^2(1-\gamma)}{\lambda\xi}E_t x_{t+1} - \beta\gamma E_t\pi_{t+1} - \beta(1-\gamma)\pi_{t-1} - u_t\right).$$

This equation may then be combined with the IS relation (8) to obtain a rule of the form

$$i_t = \hat{\alpha}^f E_t y_{t+1} + \hat{\alpha}^L y_{t-1} + \hat{\alpha}^{\hat{g}} \hat{g}_t \tag{23}$$

where

$$\hat{\alpha}^f = \left(\frac{\delta\xi(\psi+\lambda^2) - \psi\beta^2(1-\gamma)}{\xi\phi(\psi+\lambda^2)}, 1 + \frac{\lambda\beta\gamma}{\phi(\psi+\lambda^2)} \right) \tag{24}$$

$$\hat{\alpha}^L = \left(\frac{(1-\delta)(\psi+\lambda^2) - \psi\gamma\xi}{\phi(\psi+\lambda^2)}, \frac{\lambda\beta(1-\gamma)}{\phi(\psi+\lambda^2)} \right) \tag{25}$$

$$\hat{\alpha}^{\hat{g}} = \left(\frac{1}{\phi}, \frac{\lambda}{\phi(\psi+\lambda^2)} \right). \tag{26}$$

We call this the EH-rule and note that in case $\gamma = 1$, $\delta = 1$ and the timeless perspective is assumed, (23) reduces to the rule obtained by Evans and Honkapohja.

Evans and Honkapohja showed analytically that their rule always resulted in determinacy, and that the unique REE was always stable under learning. Similarly,

Proposition 3 *The reduced form model given by the IS and AS relations (8) and (9) and closed with the EH-rule (23) is determinate for both $\xi = 1$ and $\xi = \beta$.*

The proof of this proposition follows from the observation that solutions to the system (8), (9) and (23) are bijective with those of (9) and the relevant FOC. This proposition extends the determinacy result of Evans and Honkapohja to models with inertia, and to rules implementing either the timeless perspective or the MJB-alternative. Analytic results on stability are not obtainable for us, but numerically, we find the same result as EH. In particular, for all permutations of calibrations and inertial specifications, and under both the timeless perspective and the MJB-alternative, the EH-rule results in stability.

4.3 The Optimal Policy Manifold

Equations (22) and (23) indicate that there are at least two rules of the form

$$i_t = \alpha^f E_t y_{t+1} + \alpha^L y_{t-1} + \alpha^{\hat{g}} \hat{g}_t \tag{27}$$

consistent with the optimal REE (21), and further show that rules implementing the optimal REE may not impart the same stability and determinacy properties. We are then led to wonder if the collection of all rules of the form (27) can be classified, and how their associated stability and determinacy properties may vary. We turn to these issues now.

Denote by Ω the set of all $\alpha = (\alpha^f, \alpha^L, \alpha^{\hat{g}})'$ so that (27) is consistent with (21). We call Ω the optimal policy manifold, and note that it is a subset of $\mathbb{R}^6$, and furthermore is non-empty, as it contains both the fundamentals rule (22) and the EH-rule (23). Now define the matrix $\hat{c} = (0, c)$, so that the optimal REE may be written $y_t = by_{t-1} + \hat{c}\hat{g}_t$. Forming expectations with respect to this REE and imposing them into (27) yields

$$i_t = (\alpha^f b^2 + \alpha^L)y_{t-1} + (\alpha^{\hat{g}} + \alpha^f (b\hat{c} + \hat{c}\rho)\, \hat{g}_t,$$

which must be the same as the fundamentals rule. Thus $\alpha \in \Omega$ provided

$$\alpha^f b^2 + \alpha^L = \tilde{\alpha}^L \quad (28)$$

$$\alpha^f (b\hat{c} + \hat{c}\rho) + \alpha^{\hat{g}} = \tilde{\alpha}^{\hat{g}}. \quad (29)$$

Equations (28) and (29) characterize the optimal policy manifold. Furthermore, notice that we may trivially solve (28) and (29) for α^L and $\alpha^{\hat{g}}$ as linear functions of α^f. Thus the optimal policy manifold is a two-dimensional hyperplane in $\mathbb{R}^6$, given by

$$\Omega = \left\{ \alpha \in \mathbb{R}^6 : \exists \alpha^f \in \mathbb{R}^2 \text{ with } \alpha^L = \tilde{\alpha}^L - \alpha^f b^2 \text{ and } \alpha^{\hat{g}} = \tilde{\alpha}^{\hat{g}} - \alpha^f (b\hat{c} + \hat{c}\rho) \right\}.$$

The stability and determinacy properties of rules associated to elements of Ω can be characterized numerically. As an example, consider Figure 3. Here the Woodford calibration is used together with the Lag inertial specification, $\beta = .98$, $\psi = 10$ and the timeless perspective is assumed. To create the figure, a $(-1, 2) \times (-1, 2)$ lattice was imposed over the $(\alpha_\pi^f, \alpha_x^f)$-space, and at each point on the lattice, the associated element of Ω was determined and the stability and determinacy properties were recorded.[14] The center of the large gray dot indicates the location of the EH-rule, which lies in the interior of the stable determinate region.

Figure 3 Here

[14]For some points in the region of UI, there were an insufficient number of real eigenvalues for a CF-representation to exist. For details on this and other issues regarding common factor representations in monetary models, see Evans and McGough (2005a).

We have chosen this admittedly extreme case to emphasize that while the EH-rule necessarily lies in the region of stable determinacy, it may be very near a boundary and surrounded by all types of bad outcomes: unstable determinacy and stable and unstable indeterminacy are all possibilities. We conducted similar analysis on a $(-5,5)\times(-5,5)$ lattice in $(\alpha^f_\pi, \alpha^f_x)$-space for all combinations of calibration, inertia specification, β-value, ψ-value, and FOC-type, and we found the results of Figure 1 to be qualitatively universal, with the caveat that the MN-calibration with various permutations of the remaining parameters may yield relatively large regions of stable determinacy, and may house the EH-rule far from any problem region.

Analysis of the policy manifold indicates that arbitrarily choosing an optimal interest-rate rule among those available in Ω is unwise, for instability or indeterminacy may result. These results also suggest that relying on an unconstrained numerical algorithm to search for the optimal rule is ill-advised for such an algorithm will be unable to distinguish between points on the manifold – all points yield the same value of the government's objective. These searches must be constrained to those regions in policy space corresponding to stable determinacy.[15]

4.4 Model Uncertainty

While analysis of the policy manifold indicates potential problems for optimal interest-rate rules, one may wonder about the relevance of these concerns given the existence of the EH-rule, which yields stable determinacy for all calibrations. And indeed if the structural parameters of the AS and IS relations are known with precision, the EH-rule can be implemented and no concern over potential indeterminacy or instability problems is warranted. On the other hand, model uncertainty, which here takes the form of uncertainty about the true values of the model's structural parameters, may imply bad outcomes even when the EH-rule is employed. To argue this point, notice that the set of all policy rules (27) may be identified with $\mathbb{R}^6$. The optimal policy manifold is precisely the subset of $\mathbb{R}^6$ coinciding with those rules which implement the relevant FOC. A simple "continuity of eigenvalues" argument shows that since the optimal policy manifold has subsets corresponding to instability, indeterminacy, and stable sunspots, which are non-empty and

[15] This point was emphasized for other interest-rate rules in Evans and McGough (2005b).

open in the *relative* topology, there must be corresponding non-empty open subsets of $\mathbb{R}^6$ whose associated policies induce indeterminacy, instability or stable sunspots. Now suppose a policy maker chooses to use the EH-rule by estimating the structural parameters of the model and setting policy accordingly. The policy maker thinks she is choosing a point on the optimal policy manifold and in a region corresponding to stable determinacy. However, if her estimates of the structural parameters are off, it is very likely she is not on the manifold, and, much more importantly, because the location of the open sets in $\mathbb{R}^6$ corresponding to stable determinacy are not where she thinks they are, the associated model may be unstable, indeterminate, or have stable sunspot equilibria. As a concrete example of this phenomenon, we note that if the policy maker with $\psi = .1$ thinks the MN calibration with medium lag prevails and sets policy according to the associated EH-rule under the timeless perspective, but if in fact the true calibration is W or CGG then the model may exhibit either unstable indeterminacy, stable indeterminacy, or explosiveness depending on the inertial specification.

These problems indicate that, in case of model uncertainty, the optimal rule should be chosen to have nice stability and determinacy properties across possible model specifications, as well as to maximize some measure of welfare. A technique for determining these types of rules is provided in Evans and McGough (2005b).

5 Conclusion

To avoid the problem of time-inconsistency, Woodford offers the timeless perspective, which induces an invariant linear restriction characterizing the associated "optimal" REE. Jensen and McCallum, and separately, Blake, argue that assuming the commitment technology implicit in the timeless perspective, an alternate linear restriction, which we call the MJB-alternative, is available that provides, on average, a smaller loss. We extend the results of these authors to more realistic models that include lags in their structural equations, and find that while the improvement over the timeless perspective provided by the MJB alternative may be quite large, inertia in the model may mitigate this effect.

When constructing the timeless perspective and the MJB-alternative, the policy instrument is somewhat unrealistically taken to be inflation. In the second part of the paper, we then turn to issues of implementing these first order conditions by deriving appropriate interest-rate rules. We extend the results of Evans and Honkapohja to find an expectations based rule, which we call the EH-rule, that always produces a stable and determinate outcome. However, in characterizing all possible optimal policy rules, we find that many are associated to instability, indeterminacy, or stable sunspots. These findings suggest that in the presence of estimation error, bad outcomes may obtain even in case the EH-rule is employed. This serves as a strong caution to policymakers and suggests policy that is robust to parameter uncertainty and other types of model uncertainty is important.

6 Appendix

Fully Optimal FOC

For generality, assume the government's discount factor is $\hat{\delta}$, and thus may be different from private agents' discount factor (we will use this generality when proving Proposition 1 below). Set

$$R_t = \beta\left(\gamma E_t \pi_{t+1} + (1-\gamma)\pi_{t-1}\right) + \lambda x_t + u_t.$$

Ignoring expectations for the moment, the Lagrangian for the government's problem may be written

$$\mathcal{L} = \sum_{t=0}^{\infty} \hat{\delta}^t \left(\left(\psi x_t^2 + \pi_t^2\right) + \omega_t(\pi_t - R_t)\right), \tag{30}$$

where ω_t is the associated sequence of Lagrange multipliers. Differentiation yields

$$\begin{aligned}
\mathcal{L}_{\pi_t} &= 2\hat{\delta}^t \pi_t + \hat{\delta}^t \omega_t - \hat{\delta}^{t+1}\beta(1-\gamma)\omega_{t+1} - \hat{\delta}^{t-1}\beta\gamma\omega_{t-1} \text{ for } t \geq 1 && (31)\\
\mathcal{L}_{\pi_0} &= 2\pi_0 - \omega_0 + \beta(1-\gamma)\hat{\delta}\omega_1, && (32)\\
\mathcal{L}_{x_t} &= 2\hat{\delta}^t \psi x_t - \hat{\delta}^t \lambda \omega_t. && (33)
\end{aligned}$$

Setting these equations equal to zero, incorporating expectations and setting $\hat{\delta} = \beta$ yields the equations in the text.

Proof of Proposition 1

We begin with three lemmas.

Lemma 4 *For $a \in (0,1)$, let $f_n, f : [a,1] \to \mathbb{R}$ be continuous, and assume f_n converges to f uniformly. Then*

$$\lim_{x \to 1^-} (1-x) \sum_{n \geq 0} x^n f_n(x) = f(1). \tag{34}$$

Proof. Define $S : [a,1) \to \mathbb{R}$ by $S(x) = (1-x)\sum_{n\geq 0} x^n f_n(x)$. Notice that $\{f_n\}$ is uniformly bounded (sup-norm) so that the sum is absolutely convergent. In particular, S is well-defined and continuous. That the sum is absolutely convergent allows us to rearrange terms, thus yielding the first equality in the following displayed system:

$$\begin{aligned} |S(x) - f(1)| &= \left|(1-x)\sum_{n\geq 0} x^n (f_n(x) - f(1))\right| \\ &\leq (1-x)\sum_{n=0}^{M} x^n \left|(f_n(x) - f(1))\right| + (1-x)\sum_{n>M} x^n \left|(f_n(x) - f(1))\right|. \end{aligned} \tag{35}$$

Now let $\varepsilon > 0$ and choose M so that $n > M \Rightarrow |f_n(x) - f(x)| < \varepsilon/4$ for all x (by uniform convergence). Now choose $\delta_1 \in (0,1)$ so that $x \in (\delta_1, 1)$ implies $|f(x) - f(1)| < \varepsilon/4$. Then

$$(1-x)\sum_{n>M} x^n \left|(f_n(x) - f(1))\right| < \frac{\varepsilon}{2},$$

for all $x \in (\delta_1, 1)$. Now notice that

$$\lim_{x\to 1}(1-x)\sum_{n=0}^{M} x^n \left|(f_n(x) - f(1))\right| = 0,$$

so we may choose $\delta > \delta_1$ so that $x \in (\delta, 1)$ implies

$$(1-x)\sum_{n=0}^{M} x^n \left|(f_n(x) - f(1))\right| < \frac{\varepsilon}{2},$$

which completes the proof. ■

Lemma 5 *Let $a \in (a,1)$. For all $\delta \in [a,1]$, let $y_t(\delta)$ be a stationary VAR(1) process given by*

$$y_t = A(\delta)y_{t-1} + \varepsilon_t, \tag{36}$$

where $A : [a,1] \to \mathbb{R}^{n\times n}$ is continuous and ε_t iid. Pick arbitrary $\alpha \in \mathbb{R}^n$. For $t > 0$ let

$$\begin{aligned} V_t(\delta) &= E\left(y_{it}(\delta)^2 | y_0(\delta) = \alpha\right) & (37) \\ V(\delta) &= E\left(y_{it}^2\right). & (38) \end{aligned}$$

Then $V_t \to V$ uniformly on $[a,1]$.

Proof. We prove this in case $n = 1$, but the extension to the general case is clear. Writing the process $y_t(\delta)| (y_0(\delta) = \alpha)$ in terms of the initial condition and subsequent shocks, and computing expectations of the second moments yields

$$V_t(\delta) = \alpha^2 \left(A(\delta)^2\right)^t + \sigma_\varepsilon^2 \left(\frac{1 - \left(A(\delta)^2\right)^t}{1 - A(\delta)^2} \right). \tag{39}$$

Fact 1: Notice that $\left(A(\delta)^2\right)^t \to 0$ uniformly. Indeed, by continuity of A and compactness of $[a, 1]$, as well as the assumed stationarity of $y_t(\delta)$, we have that

$$\sup_{\delta \in [a,1]} A(\delta)^2 = M < 1.$$

Thus for all $\delta \in [a, 1]$, $\left(A(\delta)^2\right)^t < M^t \to 0$.

Fact 2: Notice that if $h_t, h, g : [a, 1] \to \mathbb{R}$ are continuous and $h_t \to h$ uniformly then $f_t = g \cdot h_t$ converges to $g \cdot h$ uniformly. Indeed, let $\varepsilon > 0$ and $N = \sup |g(x)| < \infty$. Choosing T so that $t \geq T$ implies $|h_t - h| < \varepsilon/N$ completes this argument.

We may now use Facts 1 and 2 to complete the proof. Let $k_t(\delta) = \alpha^2 \left(A(\delta)^2\right)^t$, $g(\delta) = \sigma_\varepsilon^2 \left(1 - A(\delta)^2\right)^{-1}$, and $h_t(\delta) = \left(A(\delta)^2\right)^t$. Then

$$V_t(\delta) = k_t(\delta) - g(\delta)h_t(\delta) + g(\delta). \tag{40}$$

But k_t and h_t converge uniformly to zero and $g(\delta) = V(\delta)$, so the proof is complete. ■

Lemma 6 *The MJB-alternative and the AS relation yield a determinate system.*

Proof. Using (31) - (33), we obtain the following FOC, which, for $\hat{\delta} < 1$, corresponds to the timeless perspective in case the government's discount $\hat{\delta}$ potentially differs from the discount factor of private agents:

$$\lambda \pi_t = -\psi(x_t - \gamma \frac{\beta}{\hat{\delta}} x_{t-1} - (1-\gamma)\beta\hat{\delta} E_t x_{t+1}). \tag{41}$$

With $\hat{\delta} < 1$, this equation, together with (46) is determinate (see Woodford (2003)). Our goal is to show that this system is determinate for $\hat{\delta} = 1$. To do this, we proceed by contradiction: assume the system is not determinate. To obtain our contradiction we require three steps.

Step 1. *At least three of the relevant eigenvalues have norm greater than or equal to one.*

To argue step 1, first notice we may assume the model is non-stochastic and that agents have perfect foresight. The FOC and AS relations may then be combined to eliminate the dependence on inflation, resulting in the following discrete dynamic system:

$$x_{t+2} - Ax_{t+1} + Bx_t - Ax_{t-1} + x_{t-2} = 0, \tag{42}$$

where

$$A = ((1-\gamma)\beta\gamma)^{-1} \text{ and } B = \frac{A}{\frac{\psi}{\lambda}\beta}\left(\lambda + \frac{\psi}{\lambda} + \frac{\psi}{\lambda}\gamma^2\beta^2 + \frac{\psi}{\lambda}(1-\gamma)^2\beta^2\right).$$

Now notice that

$$(z-\mu_1)(z-\frac{1}{\mu_1})(z-\mu_2)(z-\frac{1}{\mu_2}) = z^4 - Az^3 + Bz^2 - Az + 1,$$

where

$$A = -\left(\frac{\mu_1^2+1}{\mu_1} + \frac{\mu_2^2+1}{\mu_2}\right) \text{ and } B = \left(2 + \left(\frac{\mu_1^2+1}{\mu_1}\right)\left(\frac{\mu_2^2+1}{\mu_2}\right)\right),$$

which demonstrates that the roots of the characteristic polynomial associated to our system (42) have the form μ_1, μ_2 and their reciprocals. Step one is proved if at least three of these roots have norm less than or equal to one. So assume at least two roots have norm larger than one. Because we have assumed our model is not determinate, it must be that at least three roots have norm larger than or equal to one (two roots outside the unit circles and two roots inside the unit circle would imply determinacy). So either there exists i, so that $|\mu_i| > 1$ and $|1/\mu_i| \geq 1$ or there exists i such that $|\mu_i| \geq 1$ and $|1/\mu_i| > 1$: either implication is a contradiction.

Step 2. *Let $\hat{\delta} < 1$ and assume the x_t, π_t time series is generated by the unique solution to (46) and (41) when $\rho = 0$. Set $W(\hat{\delta}) = E(\psi x_t^2 + \pi_t^2)$. Then for any $M > 0$ there exists $\bar{\delta} \in (0,1)$ so that $\hat{\delta} \in (\bar{\delta}, 1)$ implies $W(\hat{\delta}) > M$.*

The idea behind this step is straightforward: step one shows that, under our maintained assumption, the MJB alternative yields an explosive system. Here we exploit the fact that the MJB alternative is the limit of the timeless perspective as $\hat{\delta} \to 1$ to show that, again under our maintained assumption, the variances implied by the timeless perspective become large as $\hat{\delta}$ gets near one.

Equations (46) and (41) give rise to the timeless perspective; write its unique stationary solution as

$$y_t = A(\hat{\delta})y_{t-1} + B(\hat{\delta})\varepsilon_t. \tag{43}$$

Denote by $\Lambda_i(\hat{\delta})$ the eigenvalues of $A(\hat{\delta})$. By step 1 and the stationarity of (43) for all $\hat{\delta} < 1$, for all $\epsilon > 0$ there exists a $\bar{\delta}$ so that $\hat{\delta} \in (\bar{\delta}, 1) \Rightarrow |\Lambda_i(\hat{\delta})| > 1 - \epsilon$, for at least one of the Λ_i, which we assume is Λ_1 without loss of generality. Decompose A as $S(\Lambda_1 \oplus \Lambda_2)S^{-1}$, and set $z = S^{-1}y$. Then (43) becomes

$$z_t = (\Lambda_1(\hat{\delta}) \oplus \Lambda_2(\hat{\delta}))z_{t-1} + \hat{B}(\hat{\delta})\varepsilon_t, \tag{44}$$

where $\hat{B}(\hat{\delta}) = S^{-1}B(\hat{\delta})$.

We claim that either $|\Lambda_2(\hat{\delta})| \to 1$ as $\hat{\delta} \to 1$ or that $\hat{B}_1(1) \neq 0$. To demonstrate this, we proceed by contradiction. Assume $|\Lambda_2(1)| < 1$ and $\hat{B}_1(1) = 0$. Then, for $\hat{\delta} = 1$, there is a stationary solution to (44) is given by $z_{1t} = 0$ and $z_{2t} = \Lambda_2(\hat{\delta})z_{2t-1} + \hat{B}_2(\hat{\delta})\varepsilon_t$. But this implies the existence of a stationary solution to (46) and (41), which contradicts step one, which showed any such solution is explosive.

We also claim that $\hat{B}(1) \neq 0$. Indeed, if it does then, for $\hat{\delta} = 1$, there is a stationary solution to (44) is given by $z_t = 0$, which implies the existence of a stationary solution to (46) and (41): again, a contradiction.

The two claims above show that there is an i such that $\Lambda_i(\hat{\delta}) \to 1$ and $\hat{B}_i(1) \neq 0$. For simplicity, assume $i = 1$. Because $\hat{B}(\hat{\delta})$ is continuous in $\hat{\delta}$, there is a $\hat{\delta}_1$ so that $\hat{\delta} \in (\hat{\delta}_1, 1]$ implies $|\hat{B}_1(\hat{\delta})| > \frac{1}{2}|\hat{B}_1(1)| \equiv N$. Then $\hat{\delta} \in (\hat{\delta}_1, 1)$ implies

$$var(z_{1t}) > \frac{N^2\sigma_\varepsilon^2}{1 - \Lambda_1(\hat{\delta})} \to \infty \text{ as } \hat{\delta} \to 1.$$

Now, as a brief aside, consider three random variables a, b, c with $c = a + b$, and assume a and b have finite first and second moments. Then

$$\begin{aligned} Ec^2 &\leq E(a^2+b^2)+2|Eab| \\ &\leq E(a^2+b^2)+2E|ab| \\ &\leq E(a^2+b^2)+2\left(Ea^2\right)^{\frac{1}{2}}\left(Eb^2\right)^{\frac{1}{2}}, \end{aligned}$$

where the first line is from the triangle inequality, the second is from Jensen's inequality, and the third is from the Cauchy-Schwartz inequality. We conclude that $var(z_{1t}) \to \infty$ as $\hat{\delta} \to 1$ implies that $E(x_t^2) \to \infty$ or $E(y_t^2) \to \infty$ as $\hat{\delta} \to 1$, which completes the proof of step 2.

Step 3. *Step 2 contradicts the full optimality of the timeless perspective when coupled with the FOC corresponding to initial conditions, i.e.*

$$\lambda\pi_0 = -\psi(x_0 - (1-\gamma)\beta\hat{\delta}E_0 x_1). \tag{45}$$

To obtain a contradiction, which is thus a contradiction of the maintained assumption, pick $\hat{\delta}' \in (0,1)$ and let $x_t(\hat{\delta}')$ and $\pi_t(\hat{\delta}')$ be determined by (45), (41), and (46). For arbitrary $\hat{\delta} \in (0,1)$, let

$$M(\hat{\delta}) = (1-\hat{\delta})E_0 \sum_{t\geq 0} \hat{\delta}^t(\psi x_t(\hat{\delta}')^2 + \pi_t(\hat{\delta}')^2).$$

By Lemma 4, M is well-defined and continuous on $[0,1]$, and so is uniformly bounded by some value M^*. Now choose $\bar{\delta}$ by step 2 associated to the value $2M^*$. Set

$$W_n(\hat{\delta}) = (1-\hat{\delta})E_0 \sum_{t\geq n} \hat{\delta}^t(\psi x_t(\hat{\delta})^2 + \pi_t(\hat{\delta})^2),$$

where, again, $x_t(\hat{\delta})$ and $\pi_t(\hat{\delta})$ are determined by (45), (41), and (46). Notice that the fully optimal solution corresponds to the timeless perspective under the initial condition $x_{-1} = 0$. Thus, by Lemma 5, for $\hat{\delta} \in (\bar{\delta}, 1)$

$$E_0(\psi x_t(\hat{\delta})^2 + \pi_t(\hat{\delta})^2) \to E(\psi x_t(\hat{\delta})^2 + \pi_t(\hat{\delta})^2) > 2M^*$$

uniformly as $t \to \infty$. Pick N so that $t > N$ implies $E_0(\psi x_t(\hat{\delta})^2 + \pi_t(\hat{\delta})^2) > \frac{3}{2}M^*$ for all $\hat{\delta} \in (\bar{\delta}, 1)$. Then $W_0(\hat{\delta}) \geq W_n(\hat{\delta}) > \frac{3}{2}M^*\hat{\delta}^n \to \frac{3}{2}M^*$ as $\hat{\delta} \to 1$. Thus for large enough $\hat{\delta}$, $W_0(\hat{\delta}) > M^*$. But this means that (45), (41), and (46) can not be optimal for the given $\hat{\delta}$ because policymakers could always implement the policy that yielded $x_t(\hat{\delta}')$, $\pi_t(\hat{\delta}')$ and thus an objective value at least as low as M^*. This contradiction completes step 3.■

We are now ready to attack the main problem, which we restate for clarity. Let Θ be the collection of all $\theta \in \mathbb{R}^3$ so that the system

$$\pi_t = \beta\gamma E_t \pi_{t+1} + \beta(1-\gamma)\pi_{t-1} + \lambda x_t + u_t \tag{46}$$

$$\pi_t = \theta_0 x_{t-1} + \theta_1 x_t + \theta_2 E_t x_{t+1} \tag{47}$$

is determinate. We want to find $\theta \in \Theta$ to minimize $E\pi_t^2 + \psi E x_t^2$.

To this end, we consider an alternate problem. For given initial condition $z_0 = (x_0, \pi_0, u_0)$, and for $\delta < 1$,

$$\min_{\pi_t, x_t} (1-\delta) E \sum_{t \geq 1} \delta^t (\pi_t + \psi x_t^2) \tag{48}$$

so that (46) holds. Applying Lagrange's method as usual yields the following FOC:

$$\lambda \pi_t = -\psi\left(x_t - \frac{\gamma\beta}{\delta} x_{t-1} - (1-\gamma)\beta\delta E_t x_{t+1}\right), \text{ for } t > 1 \tag{49}$$

$$\lambda \pi_1 = -\psi(x_1 - (1-\gamma)\beta\delta E_1 x_2). \tag{50}$$

For fixed δ, call (49) "$R(\delta)$" (i.e. the "Rule" associated to the discount rate δ). While we have analyzed the problem (48) only for $\delta < 1$, we may still consider the rule $R(1)$, which we note is the MJB-alternative. Recall that Woodford shows his timeless perspective, when combined with an AS relation like (46), results in a determinate model, thus the system (46) and $R(\delta)$ is determinate for $\delta < 1$. Also, according to Lemma 6, $R(1)$, together with (46) is determinate.

We claim that $y_t(1)$ is the desired solution, that is, the time path of π and x minimizing $E\pi_t^2 + \psi E x_t^2$ subject to (46) and (47). To see this let $\hat{y}_t$ be any process obtained by solving (46) and (47) for some $\theta \in \Theta$. Recall $y_t(\delta) = (x_t(\delta), \pi_t(\delta))'$. Write

$$V(\delta) = E\left(\pi_t(\delta)^2 + \psi x_t(\delta)^2\right) \tag{51}$$

$$V_t(\delta) = E\left(\pi_t(\delta)^2 + \psi x_t(\delta)^2 | z_0\right) \tag{52}$$

$$\hat{V}(\delta) = E\left(\hat{\pi}_t^2 + \psi \hat{x}_t^2\right) \tag{53}$$

$$\hat{V}_t(\delta) = E\left(\hat{\pi}_t^2 + \psi \hat{x}_t^2 | z_0\right) \tag{54}$$

Noticing that the solution $y_t(\delta)$ can be stacked with the u_t to be in VAR(1) form with damping matrix continuous in δ, we may conclude using Lemma 5 that V_t converges to V uniformly. Also, the dependence of $\hat{V}$ and $\hat{V}_t$ on δ is trivial, so convergence of $\hat{V}_t$ to $\hat{V}$ is trivially uniform.

Now assume $x_0 = 0$. Then the FOCs (49) and (50) coincide. This means that for $\delta < 1$, the conditional process $y_t(\delta)|z_0$ solves the problem (48). In particular,

$$(1-\delta)\sum_{t\geq 1}\delta^t V_t(\delta) \leq (1-\delta)\sum_{t\geq 1}\delta^t \hat{V}_t(\delta). \tag{55}$$

Now let $\delta \to 1$ and apply the lemmas to get that $V(1) \leq \hat{V}(1)$, which is precisely what we wanted to prove. ■.

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Figure 1: MJB/TP Improvement for Varying γ
Woodford calibration, ψ=1

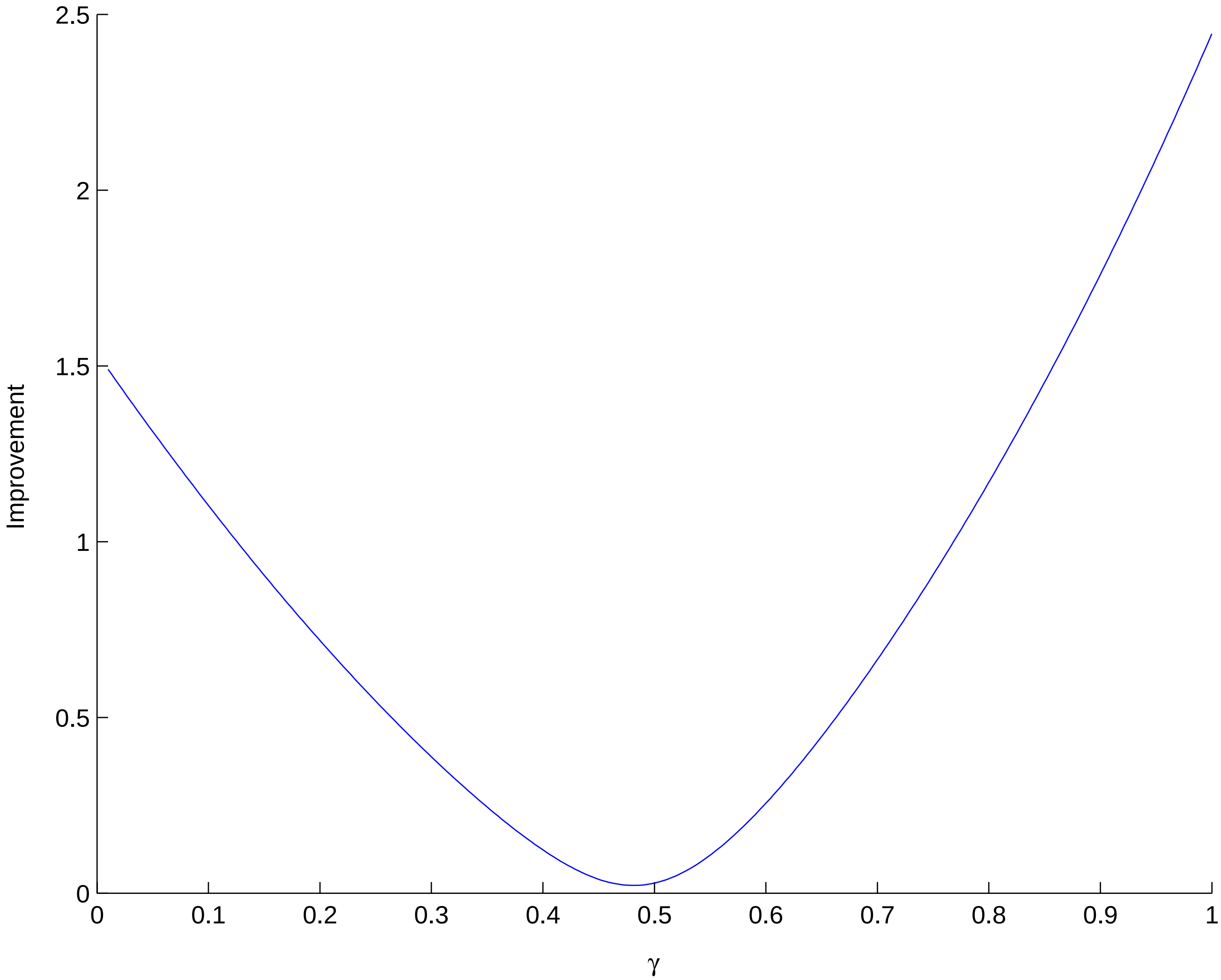

Figure 2: MJB/TP Improvement:
Woodford calibration, ψ=1

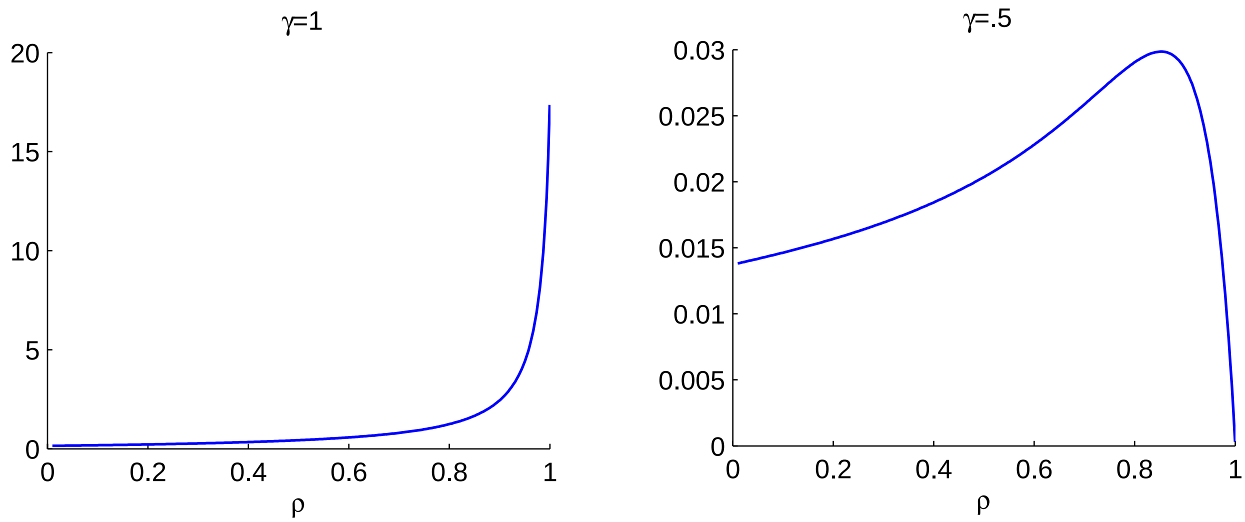

Figure 3: Policy Manifold, Timeless Perspective
ϕ=6.37, λ=.024, γ=.5, δ=.5, β=.98, ψ=10