Lesson 7

The Collective Model and the Fermi Gas Model
Evidence for Nuclear Collective Behavior

• Existence of permanently deformed nuclei, giving rise to “collective excitations”, such as rotation and vibration
• Systematics of low lying 2+ states in many nuclei
• Large transition probabilities for $2+ \rightarrow 0+$ transitions in deformed nuclei
• Existence of giant multipole excitations in nuclei
Fig. 5.8. Regions where deformed nuclei are expected are marked with circles. The diagonal line is the line of stability. Shaded areas mark regions of known deformed nuclei.
Deformed Nuclei

- Which nuclei? $A=150-190, A>220$
- What shapes?
How do we describe these shapes?

\[ R(\theta, \phi) = R_{\text{avg}} \left[ 1 + \beta Y_{20}(\theta, \phi) \right] \]

\[ \beta = \frac{4}{3} \sqrt{\frac{\pi}{5}} \frac{b - a}{R_{\text{avg}}} \]

\[ R_{\text{avg}} = \frac{1}{2} \left( a^2 + b^2 \right) \]
Rotational Excitations

• Basic picture is that of a rigid rotor

\[ E_{\text{rot}} = \frac{J(J+1)\hbar^2}{2\mathcal{I}} \]

where

\[ \mathcal{I}_{\text{sphere}} = \frac{2}{5} mR^2 \]

\[ \mathcal{I}_{\text{ellipsoid}} = \frac{2}{5} mR^2 \frac{1}{(1+0.31\beta)} \]

<table>
<thead>
<tr>
<th>\hbar^2 \text{values}</th>
<th>16+</th>
<th>14+</th>
<th>12+</th>
<th>10+</th>
<th>8+</th>
<th>6+</th>
<th>4+</th>
<th>2+</th>
<th>0+</th>
</tr>
</thead>
<tbody>
<tr>
<td>\mathcal{I}_{\text{sphere}}</td>
<td>4.14</td>
<td>3.50</td>
<td>2.88</td>
<td>2.30</td>
<td>1.75</td>
<td>1.23</td>
<td>0.76</td>
<td>0.34</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Do real nuclei follow this simple model?
Are nuclei rigid rotors?

- No, $\mathcal{J}_{\text{irrot}} < \mathcal{J}_{\text{exp}} < \mathcal{J}_{\text{rigid}}$
Backbending
Aside on super-deformed nuclei

- $a/c=2$
Gamma ray spectra

\[ E_{\text{rot}} = \frac{J(J+1)\hbar^2}{2\mathfrak{S}} \]

\[ \Delta E_{\text{rot}} (J+2 \rightarrow J) = \frac{(2J+3)\hbar^2}{\mathfrak{S}} = E_\gamma \]

\[ \Delta E_\gamma = \frac{4\hbar^2}{\mathfrak{S}} \]
Fig. 12.18. The gamma-ray spectrum of the superdeformed band in $^{152}$Dy as originally identified in the 1986 Daresbury experiment (from Twin et al., 1986).
Fig. 12.19. The full spectrum observed for the nucleus $^{152}$Dy showing the low-spin non-collective yrast states in the middle, a collective normal-deformed band to the left and the superdeformed band to the right. The inset in the upper left corner shows $E$ versus $I$ plotted in a schematic way for the different structures (from J.F. Sharpey-Schafer, Physics World, Sept. 1990, p. 31).
Rotations in Odd A nuclei

• The formula on the previous slide dealt with rotational excitations in \( e-e \) nuclei.
• What about odd \( A \) nuclei?
• Here one has the complication of coupling the angular momentum of the rotational motion and the angular momentum of the odd nucleon.
Rotations in Odd $A$ nuclei (cont.)

For $K \neq 1/2$

$$E(I) = \frac{\hbar^2}{2S} [I(I + 1) - K(K + 1)]$$

$I = K, K+1, K+2, ...$

For $K = 1/2$

$$E_{K=1/2}(I) = \frac{\hbar^2}{2S} [I(I + 1) - \alpha(-1)^{I+1/2}(I + 1/2)]$$

$I = 1/2, 3/2, 5/2, ...$
Fig. 5.12. Addition of angular momenta in an odd-$A$ deformed nucleus. In (a), $\Omega$ is the projection of the total angular momentum of the odd nucleon. It is vector-added to the rotational angular momentum $R$ to give the total angular momentum $I$ whose projection on the symmetry axis is $K$. In a symmetric nucleus, $R$ is actually perpendicular to the axis as shown in (b). Therefore in this case $\Omega = K$. 
Nuclear Vibrations

• In analogy to molecules, if we have rotations, then we should also consider vibrational states

• How do we describe them?

• Most nuclei are spherical, so we base our description on vibrations of a sphere.
Vibrations about a spherical shape

Vibrations are characterized by a multipole quantum number $\lambda$ in surface parametrization:

$$R(\theta,\phi) = R_0 \left( 1 + \sum_{\lambda} \sum_{\mu=-\lambda}^{+\lambda} \alpha_{\lambda\mu} Y_{\lambda\mu}^* (\theta,\phi) \right)$$

$\lambda=2$: quadrupole vibration
Properties of spherical vibrations

- Energy spectrum:
  \[ E_{\text{vib}}(n) = \left( n + \frac{5}{2} \right) \hbar \omega, \quad n = 0,1, \ldots \]

- \( R_{42} \) energy ratio:
  \[ \frac{E_{\text{vib}}(4^+)}{E_{\text{vib}}(2^+)} = 2 \]

- E2 transitions:
  \[ B(E2; 2^+_1 \rightarrow 0^+_1) = \alpha^2 \]
  \[ B(E2; 2^+_2 \rightarrow 0^+_1) = 0 \]
  \[ B(E2; n = 2 \rightarrow n = 1) = 2\alpha^2 \]
Possible vibrational nuclei from $R_{42}$
Vibrations about a spheroidal shape

• The vibration of a shape with axial symmetry is characterized by $a_{\lambda \nu}$.
• Quadrupole oscillations
  – $\nu=0$: along the axis of symmetry ($\beta$)
  – $\nu=\pm 1$: spurious rotation
  – $\nu=\pm 2$: perpendicular to axis of symmetry ($\gamma$)
Fig. 5.15. Vibrations of spherical and deformed nuclei.
Spectrum of spheroidal vibrations
You can build rotational levels on these vibrational levels

\[ E = \frac{\hbar^2}{2I} \left[ J(J+1) - K^2 \right] \]
Example of $^{166}$Er
\begin{align*}
E(\text{MeV}) & \quad J^\pi \\
4.038 & \quad 3^- \quad \text{Octupole vibration} \\
2.505 & \quad 4^+ \\
2.286 & \quad 0^+ \\
2.159 & \quad 2^+ \\
\{ & \text{Two phonons} \\
1.332 & \quad 2^+ \quad \text{One phonon} \\
0 & \quad 0^+ \quad \text{Spherical ground state} \\
^60\text{Ni} & 
\end{align*}

\textbf{Fig. 5.16.} Vibrational levels in the spherical nucleus $^60\text{Ni}$. 
<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>Quantum Number</th>
</tr>
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<tbody>
<tr>
<td>1.615</td>
<td>10+</td>
</tr>
<tr>
<td>1.552</td>
<td>5+</td>
</tr>
<tr>
<td>1.395</td>
<td>4+</td>
</tr>
<tr>
<td>1.367</td>
<td>6+</td>
</tr>
<tr>
<td>1.235</td>
<td>3+</td>
</tr>
<tr>
<td>1.086</td>
<td>2+</td>
</tr>
<tr>
<td>1.057</td>
<td>4+</td>
</tr>
<tr>
<td>0.905</td>
<td>6+</td>
</tr>
<tr>
<td>0.811</td>
<td>2+</td>
</tr>
<tr>
<td>0.685</td>
<td>0+</td>
</tr>
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</table>

$\gamma$-band
$K=2$

$\beta$-band
$K=0$

0.367       | 4+             |
0.122       | 2+             |
0          | 0+             |

$K=0$ band
based on
ground state

${}^{152}\text{Sm}$

**Fig. 5.17.** Ground state, $\beta$- and $\gamma$-vibrational bands in the deformed nucleus $^{152}\text{Sm}$. 
Fig. 5.18. Levels of $^{63}\text{Cu}$ formed from $2^+$ vibration of $^{62}\text{Ni}$ plus the $p_{3/2}$ odd proton.
Net Result
Nilsson model

• Build a shell model on a deformed h.o. potential instead of a spherical potential
Fig. 5.14. Spectrum of energy levels of $^{25}\text{Al}$. Each level is labelled by its energy in MeV and its spin and parity.
Successes of Nilsson model

<table>
<thead>
<tr>
<th>Nuclide</th>
<th>Z</th>
<th>N</th>
<th>Exp.</th>
<th>Shell</th>
<th>Nilsson</th>
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<tbody>
<tr>
<td>19F</td>
<td>9</td>
<td>10</td>
<td>1/2+</td>
<td>5/2+</td>
<td>1/2+</td>
</tr>
<tr>
<td>21Ne</td>
<td>10</td>
<td>11</td>
<td>3/2+</td>
<td>5/2+</td>
<td>3/2+</td>
</tr>
<tr>
<td>21Na</td>
<td>11</td>
<td>10</td>
<td>3/2+</td>
<td>5/2+</td>
<td>3/2+</td>
</tr>
<tr>
<td>23Na</td>
<td>11</td>
<td>12</td>
<td>3/2+</td>
<td>5/2+</td>
<td>3/2+</td>
</tr>
<tr>
<td>23Mg</td>
<td>12</td>
<td>11</td>
<td>3/2+</td>
<td>5/2+</td>
<td>3/2+</td>
</tr>
</tbody>
</table>
Real World Nilsson Model
Fig. 11.5. A similar single-particle diagram as in fig. 11.4 with measured band heads of a number of odd-proton rare-earth nuclei to the right. In each case, a rotational band is built on the band heads in a similar way as for $^{165}$Tm in fig. 11.4. The orbital of the ground state rotational band is indicated in each case (we are grateful to Sven Åberg who prepared this and the following figure).
Fermi Gas Model

- Treat the nucleus as a gas of non-interacting fermions
- Suitable for predicting the properties of highly excited nuclei
Details

Consider a box with dimensions $L_x, L_y, L_z$ in which particle states are characterized by their quantum numbers, $n_x, n_y, n_z$. Change our descriptive coordinates to $p_x, p_y, p_z$, or their wavenumbers $k_x, k_y, k_z$ where $k_i = n_i \pi / L_i = p_i / \hbar$.

The highest occupied level $\equiv$ Fermi level.

$$k^2 = k_x^2 + k_y^2 + k_z^2$$
$$\frac{k^2 L^2}{\pi^2} = n_x^2 + n_y^2 + n_z^2$$
$$E(n_x, n_y, n_z) = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m} (k_x^2 + k_y^2 + k_z^2) = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$$

Imposing quantization conditions

$$N_{states} = \left( \frac{1}{8} \right) \frac{4\pi}{3} \left( \frac{kL}{\pi} \right)^3$$
Making the box a nucleus

\[ k_f = \frac{\pi}{L} \left( \frac{2N_{\text{states}}}{3\pi} \right)^{1/3} = \frac{\pi}{L} \left( \frac{2Z}{3\pi} \right)^{1/3} = \frac{\pi}{r_0} \left( \frac{2Z}{3\pi A} \right)^{1/3} \]

Then

\[ E_f = \frac{k_f^2 \hbar^2}{2m} \approx 32 \text{ MeV} \]

\[ E_f^{\text{protons}} = 53 \left( \frac{Z}{A} \right)^{1/3} \text{ MeV} \]

\[ E_f^{\text{neutrons}} = 53 \left( \frac{A - Z}{A} \right)^{1/3} \text{ MeV} \]
Thermodynamics of a Fermi gas

• Start with the Boltzmann equation

\[ S = k_B \ln \Omega \]

• Applying it to nuclei

\[ S = k \ln \Omega = k \{ \ln [\rho(E^*)] - \ln [\rho(0)] \} \]

\[ S = \int_{0}^{T_e} \frac{dE^*}{T} \]

\[ E^* = a(kT)^2 \]

\[ dE^* = 2ak^2TdT \]

\[ S = 2ak^2 \int_{0}^{T_e} dT = 2ak^2T = 2k(aE^*)^{1/2} \]

\[ \frac{\rho(E^*)}{\rho(0)} = \exp \left( \frac{S}{k} \right) = \exp \left[ 2(aE^*)^{1/2} \right] \]
Relation between Temperature and Level Density

• Lang and LeCouteur

\[ \rho(E) = \frac{1}{12} \left( \frac{\pi^2}{a} \right)^{1/4} \frac{1}{(E + T)^{5/4}} \exp[2(aE)^{1/2}] \]

\[ E = aT^2 - T \]
\[ a = A/10 \]

• Ericson

\[ \rho(E) = \frac{1}{12} \left( \frac{\pi^2}{a} \right)^{1/4} \frac{1}{(E)^{5/4}} \exp[2(aE)^{1/2}] \]

\[ E = aT^2 \]
\[ a = A/10 \]
What is the utility of all this?