Lesson 11

Nuclear Reactions
Nomenclature

• Consider the reaction
  \[ {}^4\text{He} + {}^{14}\text{N} \rightarrow {}^{17}\text{O} + {}^1\text{H} \]

• Can write this as
  Projectile P + Target T \rightarrow \text{Residual Nucleus R}
  and \text{Emitted Particle x}

• Or
  \[ T(P,x)R \]
  \[ {}^{14}\text{N}(^4\text{He},p){}^{17}\text{O} \]
Conservation Laws

- Conservation of neutrons, protons and nucleons

\[ ^{59}\text{Co}(p,n)? \]

Protons \( 27 + 1 = 0 + x \)

Neutrons \( 32 + 0 = 1 + y \)

Product is \( ^{59}\text{Ni} \)
Conservation Laws (cont.)

- Conservation of energy

Consider $^{12}C(^{4}He,^{2}H)^{14}N$

$$Q = \text{(masses of reactants)} - \text{(masses of products)}$$

$$Q = M(^{12}C) + M(^{4}He) - M(^{2}H) - M^{14}N$$

$Q = +$ exothermic

$Q = -$ endothermic
Conservation Laws (cont.)

- Conservation of momentum
  \[ mv = (m+M)V \]
  \[ T_R = T_i \frac{m}{m+M} \]

- Suppose we want to observe a reaction where \( Q = - \).
  \[ -Q = T - T_R = T \frac{M}{M+m} \]
  \[ T = -Q \frac{(M+m)}{M} \]
Center of Mass System

(a) before collision
as seen in the laboratory

(b) after collision

(c) before collision
as seen from the center of mass

(d) after collision
Center of Mass System

\[ p_i = p_T \]
\[ p_{\text{tot}} = 0 \]
velocity of cm = \( V_{\text{cm}} \)
velocity of incident particle = \( v - V_{\text{cm}} \)
velocity of target nucleus = \( V_{\text{cm}} \)
\[ m(v - V_{\text{cm}}) = M V \]
\[ m(v - V_{\text{cm}}) - M V = 0 \]
\[ V_{\text{cm}} = m v / (m + M) \]
\[ T' = (1/2)m(v - V_{\text{cm}})^2 + (1/2)MV^2 \]
\[ T' = T_i * m / (m + M) \]
Center of Mass System

\[ E_T = E_1 + Q \]
\[ R = (M_1 + M_2)(M_3 + M_4) \]
\[ A = M_1 M_4 (E_1 / R E_T) \]
\[ B = M_1 M_3 (E_1 / R E_T) \]
\[ C = \frac{M_2 M_3}{R} \left( 1 + \frac{M_1 Q}{M_2 E_T} \right) \]
\[ D = \frac{M_2 M_4}{R} \left( 1 + \frac{M_1 Q}{M_2 E_T} \right) \]

If particle \( M_3 \) is observed at a laboratory angle \( \theta \):

CM energy of \( M_3 = E_T D \)

CM angle of \( M_3 = \sin^{-1} \left[ \left( \frac{E_3}{E_T D} \right)^{1/2} \sin \theta \right] \]

\[ \frac{d\sigma}{d\Omega}_{\text{CM}} = \frac{d\sigma}{d\Omega}_{\text{lab}} \frac{(AC)^{1/2}(D/B - \sin^2 \theta)^{1/2}}{E_3/E_T} \]

\[ E_3 = E_T B \left[ \cos \theta + (C/A - \sin^2 \theta)^{1/2} \right]^2 \]

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Kinematics

\[ Q = T_x \left(1 + \frac{m_x}{m_R}\right) - T_p \left(1 - \frac{m_p}{m_R}\right) - \frac{2}{m_R} \left(m_p T_p m_x T_x \right)^{1/2} \cos \theta \]

\[ T_x^{1/2} = \frac{\left(m_p m_x T_p \right)^{1/2} \cos \theta \pm \{ m_p m_x T_p \cos^2 \theta + (m_R + m_x) [m_R Q + (m_R - m_p) T_p] \}^{1/2}}{m_R + m_x} \]
Reaction Types and Mechanisms
## Reaction Types and Mechanisms

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Nuclear Reaction Cross Sections

\[ \text{fraction of beam particles that react} = \text{fraction of } A \text{ covered by nuclei} \]

\[ a \text{ (area covered by nuclei)} = n \text{ (atoms/cm}^3) \times x \text{ (cm)} \times (\sigma \text{ (effective area of one nucleus, cm}^2)) \]

\[ \text{fraction} = \frac{a}{A} = nx\sigma \]

\[ -d\phi = \phi nx\sigma \]

\[ \phi_{\text{trans}} = \phi_{\text{initial}} e^{-nx\sigma} \]

\[ \phi_{\text{initial}} - \phi_{\text{trans}} = \phi_{\text{initial}} (1 - e^{-nx\sigma}) \]
Factoids about $\sigma$

- Units of $\sigma$ are area (cm$^2$).
- Unit of area=$10^{-24}$ cm$^2$= 1 barn
- Many reactions may occur, thus we divide $\sigma$ into partial cross sections for a given process, with no implication with respect to area.
- Total cross section is sum of partial cross sections.
Differential cross sections

\[ dN/d\Omega = \phi n (d\sigma/d\Omega) dx \]

\[
\sigma = \int_0^{2\pi} \int_0^\pi \frac{d\sigma}{d\Omega}(\theta) \sin \theta d\theta d\phi
\]
Charged Particles vs Neutrons

(Number of reactions per sec) = (Number of target atoms per cm$^2$) \times \sigma \times I

In reactors, particles traveling in all directions
Number of reactions/s = Number of target atoms \times \sigma \times (\text{particles/cm}^2/\text{s})
What if the product is radioactive?

\[
\frac{dN}{dt} = (\text{rate of production}) - (\text{rate of decay})
\]

\[
\frac{dN}{dt} = n \sigma \Delta x \phi - \lambda N
\]

\[
\frac{dN}{n \sigma \Delta x \phi - \lambda N} = dt
\]

\[
\frac{d(\lambda N)}{\lambda N - n \sigma \Delta x \phi} = -\lambda dt
\]

\[
\ln(\lambda N - n \sigma \Delta x \phi) \big|_0^N = -\lambda t \big|_0^t
\]

\[
\frac{\lambda N - n \sigma \Delta x \phi}{-n \sigma \Delta x \phi} = e^{-\lambda t}
\]

\[
A = \lambda N = n \sigma \Delta x \phi (1 - e^{-\lambda t})
\]
Neutron Cross Sections-General Considerations

\[ \sigma \approx \pi (R + r')^2 = \pi r_0^2 (A_p^{1/3} + A_T^{1/3})^2 \]

\[ \ell = r \quad x \quad \vec{p} = pb \]

\[ p = \frac{\hbar}{\lambda} \]

\[ \ell \hbar = \frac{\hbar b}{\lambda} \]

\[ b = \ell \lambda \]
\[ \sigma_\ell = \pi (\ell + 1)^2 \lambda^2 - \pi \ell^2 \lambda^2 \]

\[ \sigma_\ell = \pi \lambda^2 (\ell^2 + 2\ell + 1 - \ell^2) \]

\[ \sigma_\ell = \pi \lambda^2 (2\ell + 1) \]

\[ \sigma_{\text{total}} = \sum_\ell \sigma_\ell = \sum_{\ell=0}^{\ell_{\text{max}}} \pi \lambda^2 (2\ell + 1) = \pi \lambda^2 \sum_{\ell=0}^{\ell_{\text{max}}} (2\ell + 1) = \pi \lambda^2 (\ell_{\text{max}} + 1)^2 \]
\[ l_{\text{max}} = \frac{R}{\lambda} \]

\[ l_{\text{max}} + 1 = \frac{R + \lambda}{\lambda} \]

\[ \sigma_{\text{total}} = \pi (R + \lambda)^2 \]

Semi-classical→QM

\[ \sigma_{\text{total}} = \pi \lambda^2 \sum_{\ell=0}^{\infty} (2\ell + 1) T_{\ell} \]
The transmission coefficient $T_{\ell}$

- Sharp cutoff model (higher energy neutrons)

  \[ T_{\ell} = 1 \quad \text{for} \quad \ell \leq \ell_{\text{max}} \]

  \[ T_{\ell} = 0 \quad \text{for} \quad \ell > \ell_{\text{max}} \]

- Low energy neutrons

  \[ \sigma_{\text{total}} \propto \pi \lambda^2 \sqrt{\varepsilon} \propto \pi \frac{\hbar^2}{2m\varepsilon} \sqrt{\varepsilon} \propto \frac{1}{\sqrt{\varepsilon}} \]
\[ \frac{1}{v} \]
Charged Particle Cross Sections-General Considerations

\[ B = \frac{Z_1 Z_2 e^2}{R} \]

\[ p = (2mT)^{1/2} = (2\mu)^{1/2}(\varepsilon - B)^{1/2} = (2\mu \varepsilon)^{1/2}(1 - B/\varepsilon)^{1/2} \]

reduced mass \( \mu = \frac{A_1 A_2}{A_1 + A_2} \)

\[ \ell = \overrightarrow{rxp} \]

\[ \ell_{\text{max}} = R(2\mu \varepsilon)^{1/2}\left(1 - \frac{B}{\varepsilon}\right)^{1/2} \]
Semi-classical → QM

\[ \ell \rightarrow \ell \hbar \]

\[ \sigma_{total} = \pi \hat{\lambda}^2 \left( \ell_{max} + 1 \right)^2 \approx \pi \hat{\lambda}^2 \ell_{max}^2 = \pi \hat{\lambda}^2 R^2 \frac{2\mu \varepsilon}{\hbar^2} \left( 1 - \frac{B}{\varepsilon} \right) = \pi \hat{\lambda}^2 R^2 \frac{1}{\hat{\lambda}^2} \left( 1 - \frac{B}{\varepsilon} \right) \]

\[ \sigma_{total} = \pi R^2 \left( 1 - \frac{B}{\varepsilon} \right) \]
Barriers for charged particle induced reactions

\[ V_{\text{tot}}(R) = V_C(R) + V_{\text{nucl}}(R) + V_{\text{cent}}(R) \]

\[ V_C(r) = \frac{Z_1 Z_2}{r} \quad \text{for } r < R_C \]

\[ V_C(r) = \left( \frac{Z_1 Z_2}{R_C} \right) \left( \frac{3}{2} - \frac{1}{4} \frac{r^2}{R_C^2} \right) \quad \text{for } r > R_C \]

\[ V_{\text{nucl}}(r) = V_0 / \left( 1 + \exp \left( \left( r - R/a \right) \right) \right) \]

\[ V_{\text{cent}}(r) = \frac{\hbar^2}{2\mu} \frac{\ell(\ell+1)}{r^2} \]
Rutherford Scattering

\[ F_{\text{coul}} = \frac{Z_1 Z_2 e^2}{r^2} \]

\[ PE = \frac{Z_1 Z_2 e^2}{r} \]
Rutherford Scattering
At $r=d$

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 + \frac{Z_1Z_2e^2}{d}$$

Rearranging

$$\left(\frac{v_0}{v}\right)^2 = 1 - \frac{d_0}{d}$$

where

$$d_0 = \frac{2Z_1Z_2e^2}{mv^2} = \frac{Z_1Z_2e^2}{T_p}$$

Conservation of angular momentum

$$mvb = mv_0d$$

$$b^2 = \left(\frac{v_0}{v}\right)^2d^2 = d(d - d_0)$$

Property of a hyperbola

$$d = b \cot(\varphi/2)$$

$$\tan \alpha = \frac{2b}{d_0}$$

$$\cot\left(\frac{\theta}{2}\right) = \frac{2b}{d_0}$$
Consider $I_0$ particles/unit area incident on a plane normal to the beam.
Flux of particles passing through a ring of width $db$ between $b$ and $b+db$ is
d$I = (\text{Flux/unit area})(\text{area of ring})$
d$I = I_0 \,(2\pi b \, db)$

$$dI = \frac{1}{4} \pi I_0 d_0^2 \frac{\cos\left(\frac{\theta}{2}\right)}{\sin^3\left(\frac{\theta}{2}\right)} \, d\theta$$

$$\frac{d\sigma}{d\Omega} = \frac{dI}{I_0 \, d\Omega} = \left(\frac{d_0}{4}\right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)} = \left(\frac{Z_1 Z_2 e^2}{4T_p^{cm}}\right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}$$
Removing the effect of Rutherford scattering

elastic scattering
$^{58}\text{Ni} (p, p')$

17.7 MeV

$\theta = 140^\circ$

Counts

Proton energy (arbitrary units)
Elastic \((Q=0)\) and inelastic \((Q<0)\) scattering

- To represent elastic and inelastic scattering, need to represent the nuclear potential as having a real part and an imaginary part.

\[ V = V_0 + iW_0 \]

This is called the optical model
\[
\frac{d^2 \psi}{dx^2} + \frac{2m}{\hbar^2} [E - V] \psi = 0
\]

\[
K^2 = \frac{2m}{\hbar^2} (E - V) = \frac{2m}{\hbar^2} (E - V_0 - i\omega)
\]
\[
\frac{d^2 \psi}{dx^2} + K^2 \psi = 0
\]

\[
\psi = A e^{iKx}
\]

\[
K = K_r + iK_i
\]

\[
\therefore \quad \psi = A e^{i(K_r + iK_i)x} = A e^{iK_r x} e^{-K_i x}
\]

\[
K_i \approx -\left(\frac{2m}{\hbar^2}\right)^{1/2} \frac{W_0}{\sqrt{E - V_0}}
\]
\[ V = (V_0 + iW_0 + V_s l \cdot s)[1 + e^{-\frac{(r-R)}{a}}]^{-1} \]
\[ R = R_0[1 + \beta_2 Y^0_2(\theta) + \beta_4 Y^0_4(\theta) + B_6 Y^0_6(\theta) + \cdots] \]
Direct Reactions

- Direct reactions are reactions in which one of the participants in the initial two body interaction leaves the nucleus without interacting with another particle.
- Classes of direct reactions include stripping and pickup reactions.
- Examples include (d,p), (p,d), etc.
(d,p) reactions
(d,p) reactions

\[ k_n^2 = k_d^2 + k_p^2 - 2k_d k_p \cos \theta \]

\[ \ell_n \hbar = r \times p = R k_n \hbar \]

\[ \ell_n = R k_n \]

\[ \left| \left( J_A - \ell_n \right) - \frac{1}{2} \right| \leq J_B^* \leq J_A + \ell_n + \frac{1}{2} \]

\[ \pi_A \pi_B = (-1)^\ell \]
Use of (d,p) reactions

• Surrogate for n capture
• “Negative kinetic energy neutrons”
• Spectroscopy