**Section 5.6 - Rank and Nullity.**

Read Section 5.6 Try odd numbered problems.

**Def:** For an $m \times n$ matrix $A$ the *fundamental vector spaces* associated with $A$ are the row space of $A$, the column space of $A$, the nullspace of $A$ and the nullspace of $A^T$.

**Theorem 5.6.1:** If $A$ is any matrix, then the row space and column space have the same dimension.

**Proof:**

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**Def:** The common dimension of the row space of $A$ and the column space of $A$ is called the *rank* of $A$. The *nullity* of $A$ is the dimension of the nullspace of $A$.

**Example:**

**Theorem 5.6.2:** If $A$ is any matrix, then $\text{rank}(A) = \text{rank}(A^T)$.

**Proof:**

**Theorem 5.6.3:** If $A$ is $m \times n$, then $\text{rank}(A) + \text{nullity}(A) = n$.

**Proof:**

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**Theorem 5.6.4:** If $A$ is $m \times n$,

(a) $\text{rank}(A) =$ number of variables corresponding to leading ones in the solution of $Ax = 0$.

(b) $\text{nullity}(A) =$ number of parameters in the general solution of $Ax = 0$.

**Proof:**

**Note:** $\text{rank}(A) \leq \max\{m, n\}$

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**Theorem 5.6.5:** If $Ax = b$ is a system of $m$ equations in $n$ unknowns, then TFAE

(a) $Ax = b$ is consistent

(b) $b$ is in the column space of $A$

(c) The coefficient matrix $A$ and the augmented matrix $(A|b)$ have the same rank.

**Proof:**
**Theorem 5.6.6:** If \( Ax = b \) is a system of \( m \) equations in \( n \) unknowns, then TFAE

(a) \( Ax = b \) is consistent for every \( m \) by 1 vector \( b \)
(b) The column space of \( A \) is \( \mathbb{R}^m \)
(c) \( \text{rank}(A) = n \).

**Proof:**

**Theorem 5.6.7:** If \( Ax = b \) is a consistent system of \( m \) equations in \( n \) unknowns, and if \( A \) has rank \( r \), then the general solution of the system has \( n - r \) parameters.

**Proof:**

**Theorem 5.6.8:** If \( A \) is \( m \) by \( n \), then TFAE

(a) \( Ax = 0 \) has only the trivial solution
(b) The column vectors of \( A \) are linearly independent
(c) \( Ax = b \) has at most one solution for every \( m \) by 1 vector \( b \).

**Proof:**

**Def:** Overdetermined and underdetermined systems.

**Theorem 5.6.9:** If \( A \) is \( n \) by \( n \), TFAE

1. to 9. as before
10. The columns of \( A \) are linearly independent
11. The rows of \( A \) are linearly independent
12. The columns of \( A \) span \( \mathbb{R}^n \)
13. The rows of \( A \) span \( \mathbb{R}^n \)
14. The columns of \( A \) form a basis for \( \mathbb{R}^n \)
15. The rows of \( A \) form a basis for \( \mathbb{R}^n \)
16. \( A \) has rank \( n \)
17. \( A \) has nullity 0