§ 5.3 - Linear independence

Read Section 5.3 Try odd numbered problems.

**Def:** If $S = \{v_1, \cdots, v_r\}$ is a nonempty set of vectors in a vector space $V$ and 
$$k_1v_1 + k_2v_2 + \cdots + k_rv_r = 0$$
has only the trivial solution, then $S$ is called *linearly independent*.
If there are other solutions, $S$ is called *linearly dependent*.

### Examples

- $i, j, k$ in $\mathbb{R}^3$.
- $(2,1,3), (1,2,4), (5,4,10)$ in $\mathbb{R}^3$.
- $\begin{pmatrix} 2 & 1 & 5 & 0 \\ 1 & 2 & 4 & 0 \\ 3 & 4 & 10 & 0 \end{pmatrix}$ $\sim$ $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
- $\{1, x, x^2, \cdots, x^n\}$ in $P_n$

### Theorem 5.3.1

**A set $S$ with two or more vectors is**

(a) Linearly dependent $\iff$ at least one of the vectors in $S$ can be expressed as a linear combination of the other vectors.

(b) Linearly independent $\iff$ no vector in $S$ can be expressed as a linear combination of the other vectors.

**Proof.**

### Theorem 5.3.2:

**A finite set of vectors that contains the zero vector is linearly dependent.**

(b) A set with exactly two vectors is linearly independent $\iff$ neither vector is a multiple of the other.

**Proof:**

**Geometric Determination in $\mathbb{R}^2$ and $\mathbb{R}^3$**
Theorem

Let \( S = \{v_1, \cdots, v_r\} \) be a set of vectors in \( \mathbb{R}^n \). If \( r > n \), then the set is linearly dependent.

Proof:

Def: If \( f_1(x), f_2(x), \cdots, f_n(x) \) are \( n-1 \) times differentiable functions on \( \mathbb{R} \), then the determinant of

\[
\begin{vmatrix}
    f_1(x) & f_2(x) & \cdots & f_n(x) \\
    f'_1(x) & f'_2(x) & \cdots & f'_n(x) \\
    \vdots & \vdots & \ddots & \vdots \\
    f^{(n-1)}_1(x) & f^{(n-1)}_2(x) & \cdots & f^{(n-1)}_n(x)
\end{vmatrix}
\]

is called the Wronskian of the functions and is denoted \( W(x) \).

Theorem 5.3.4 If the functions \( f_1, f_2, \cdots, f_n \) have \( n-1 \) continuous derivatives on \( \mathbb{R} \), and if \( W(x) \) is not identically 0, then the functions form a linearly independent set in \( C^{(n-1)} \).

Proof: