Section 5.2 - Subspaces, Linear Combinations

Read Section 5.2 Try odd numbered problems.

**Def:** A subset $W$ of a vector space $V$ is a *subspace of $V$* if $W$ itself is a vector space under the operations of addition and scalar multiplication defined on $V$.

**Theorem 5.2.1** Let $W$ be a nonempty subset of a vector space $V$. $W$ is a subspace if and only if the following two conditions hold:

(a) $u$ and $v$ in $W \implies u + v \in W$

(b) $k$ a scalar and $u \in W \implies ku \in W$

**Proof:**

**Examples:**

- Lines through the origin
- Symmetric or upper triangular matrices in $M_{mn}$
- Polynomials of degree $\leq n$

**Theorem 5.2.2** If $Ax = 0$ is a homogeneous system of $m$ equations in $n$ unknowns, then the solution set is a subspace of $R^n$.

**Def:** A vector $w$ is called a *linear combination* of vectors $v_1, \cdots, v_r$ if there are scalars $k_1, \cdots, v_r$ such that

$$w = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r.$$

**Examples:**

**Theorem 5.2.3:** If $v_1, \cdots, v_r$ are vectors in a vector space $V$, then

(a) The set $W$ of all linear combinations of these vectors is a subspace of $V$.

(b) $W$ is the smallest subspace of $V$ containing these vectors.

**Proof:**

**Def:** If $S = \{v_1, \cdots, v_r\}$ is a set of vectors in a vector space $V$, the the subspace $W$ of $V$ described in the previous theorem is called the *space spanned by $S$*. We say the vectors $v_1, \cdots, v_r$ span $W$. 
Notation: $W = \text{span}(S) = \text{span}(v_1, \cdots, v_r)$

Examples:

**Theorem 5.2.4** If $S = \{v_1, \cdots, v_r\}$ and $S' = \{w_1, \cdots, w_k\}$ are two sets of vectors in a vector space $V$, then $\text{span}(S) = \text{span}(S')$ if and only if each vector in $S$ is a linear combination of vectors in $S'$ and if each vector in $S'$ is a linear combination of vectors in $S$. 