**Section 4.2 - Linear Transformations**

Read Section 4.2 Try odd numbered problems.

The concept of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ is one of the central concepts in this course. Review on your own the definitions of function, domain and range on page 173.

**Def:** A function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called a map or transformation. If $m = n$, the transformation is often called an operator on $\mathbb{R}^n$.

To specify such a function, we let $x_1, \cdots, x_n$ represent the coordinates in $\mathbb{R}^n$ and we let $w_1, \cdots, w_m$ represent coordinates in $\mathbb{R}^m$. The function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is then determined by the $m$ real valued functions

\[
\begin{align*}
  w_1 &= f_1(x_1, \cdots, x_n) \\
  w_2 &= f_2(x_1, \cdots, x_n) \\
  &\vdots \\
  w_m &= f_m(x_1, \cdots, x_n)
\end{align*}
\]

We often write $f = (f_1, f_2, \cdots, f_m)$.

**Examples:**

**Def:** A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if it is defined by equations of the form

\[
\begin{align*}
  w_1 &= a_{11}x_1 + \cdots + a_{1n}x_n \\
  w_2 &= a_{21}x_1 + \cdots + a_{2n}x_n \\
  &\vdots \\
  w_m &= a_{m1}x_1 + \cdots + a_{mn}x_n
\end{align*}
\]

**Note** that this just says that each of the component functions of $T$ are linear real valued functions of the variables $x_1, \cdots, x_n$.

**Also Note** that if $A$ is the coefficient matrix, this set of equations can be rewritten in matrix form as

\[
w = Ax
\]

where $w = [w_1, \cdots, w_m]^T$ and where $x = [x_1, \cdots, x_n]^T$.

The matrix $A$ is called the standard matrix for the linear transformation $T$ and $T$ is often described as multiplication by $A$.

**Examples:**
There is a correspondence between linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $m$ by $n$ matrices $A$.

**Notation:** $T, T_A, A = [T] = [T_A]$  
**Composition:**  
If $T_A : \mathbb{R}^n \to \mathbb{R}^k$ and $T_B : \mathbb{R}^k \to \mathbb{R}^m$ are linear transformations, the transformation $T_B \circ T_A : \mathbb{R}^n \to \mathbb{R}^m$ is defined to be the transformation given by  
\[
(T_B \circ T_A)(x) = T_B(T_A(x))
\]

Note: This is one of the main reasons why matrix multiplication is defined the way that it is.

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Examples of specific types of linear transformations:

- Zero and the identity
- Reflection Operators
- Projection Operators
- Rotation Operators
- Dilation and Contraction Operators

Note: Composition is not necessarily commutative.