1.6 - Systems of Equations and Invertibility

Read 1.6 and try odd numbered problems.

**Theorem 1.6.1:** Every system of linear equations has either one solution, no solutions or infinitely many solutions.

**Proof:** We just need to show that if the system has two solutions, then it has infinitely many.

Suppose $Ax = b$ had two solutions $x_1$ and $x_2$. Let $x_0 = x_1 - x_2$. Then ⋯

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**Theorem 1.6.2** $A_{n \times n}$ is invertible if and only if for each $n \times 1$ matrix $b$, the system $Ax = b$ has exactly one solution, namely $x = A^{-1}b$.

**Proof.**

Assume $A$ is invertible. Then: ⋯

Assume for each $n \times 1$ matrix $b$, the system $Ax = b$ has exactly one solution.

Then the system $Ax = 0$ has only the trivial solution. So ⋯.

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**Examples of linear systems with the same coefficient matrix**

**Theorem 1.6.3** Let $A$ be a square matrix.

(a) If $B$ is a square matrix and $BA = I$, then $B = A^{-1}$.

(b) If $C$ is a square matrix and $AC = I$, then $C = A^{-1}$.

**Proof:**

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**Theorem 1.6.4** If $A$ is $n \times n$, then the following are equivalent

1. $A$ is invertible
2. $Ax = 0$ has only the trivial solution
3. The rref of $A$ is $I$
4. $A$ is expressible as a product of elementary matrices.
5. $Ax = b$ is consistent for each $n \times 1$ matrix $b$
6. $Ax = b$ has exactly one solution for every $n \times 1$ matrix $b$

**Proof** $(1) \implies (6) \implies (5) \implies (1)$
Theorem 1.6.5. Let $A$ and $B$ be square $n \times n$ matrices. If $AB$ is invertible, then $A$ and $B$ must also be invertible.

Proof:

Fundamental Problem:

Let $A$ be $m \times n$. Find all $m \times 1$ matrices $b$ such that $Ax = b$ is consistent.

If $A$ is invertible, we know the answer. If $A$ is either not square, or square and not invertible, we don’t.

Examples: