1.5 - Elementary Matrices

Read 1.5 and try odd numbered problems.

Def: An $n \times n$ matrix is called an elementary matrix if it can be obtained from $I_n$ by preforming a single elementary row operation.

Examples:

Theorem 1.5.1 If the elementary matrix $E$ is obtained from $I_n$ by performing a certain row operation, and if $A$ is $n \times n$, then $EA$ is the matrix obtained from $A$ by performing the same row operation.

Proof:

Note: If a row operation is performed on $I_n$ to obtain $E$, an inverse row operation can be performed on $E$ to get back to $I_n$.

Row operations on $I$ to get $E$

Multiply row $i$ by $c$

Interchange rows $i$, $j$

Add $c$ times row $i$ to row $j$

Inverse row operations on $E$ to get back to $I$

Multiply row $i$ by $1/c$

Interchange rows $i$, $j$

Add $-c$ times row $i$ to row $j$

Theorem 1.5.2 Every elementary matrix is invertible, and the inverse is also elementary.

Proof:

Theorem 1.5.3 The following statements are equivalent for an $n \times n$ matrix $A$:

1. $A$ is invertible
2. $Ax = 0$ has only the trivial solution.
3. The rref of $A$ is $I_n$
4. $A$ can be written as a product of elementary matrices.
Proof:
(1) $\Rightarrow$ (2)
(2) $\Rightarrow$ (3)
(3) $\Rightarrow$ (4)
(4) $\Rightarrow$ (1)

Def: Matrices that can be obtained from one another by a finite sequence of elementary row operations are called row equivalent.

Note: $A$ is invertible if and only if it is row equivalent to the identity matrix.

Method for finding $A^{-1}$ or for showing it doesn’t exist.
If $E_kE_{k-1}\cdots E_2E_1A = I_n$ where the $E_i$ are elementary matrices, then by multiplying both sides by $A^{-1}$ on the right, we obtain

$$E_kE_{k-1}\cdots E_2E_1 = A^{-1}$$

$$\begin{pmatrix} A & I \end{pmatrix}$$

row operations

$$\begin{pmatrix} I & A^{-1} \end{pmatrix}$$

Examples:

$$\begin{pmatrix} 2 & 3 & -1 & 1 & 0 & 0 \\ 2 & -4 & 2 & 0 & 1 & 0 \\ 3 & 7 & -3 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 2 & -\frac{3}{2} & -1 \\ 0 & 0 & 1 & \frac{13}{3} & -\frac{19}{6} & -\frac{7}{3} \end{pmatrix}$$