Quantum Mechanical Properties of Nuclei

Lecture 4
Sizes and Shapes of Nuclei

- How big are nuclei?

- 1-10 fm
- sharp cutoff model, $R = r_0 A^{1/3}$ fm

\[ r_0 = 1.07 \]
\[ r_0 = 1.2 \]
\[ r_0 = 1.44 \]
Nuclear Sizes (cont.)

- Diffuse surface model

\[
\rho(r) = \frac{\rho_0}{1 + e^{(r-R)/a}}
\]

\[
\rho_0 = 0.172 \text{nucleons}/\text{fm}^3
\]

\[
t = 4a \cdot \ln(3) \approx 4.4a \approx 2.4 - 2.5 \text{ fm}
\]

\[
t = \text{radial distance between 90\% and 10\% density points}
\]

Lighter nuclei are mostly surface
<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Fraction of nucleons in the &quot;skin&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{12}$C</td>
<td>0.90</td>
</tr>
<tr>
<td>$^{24}$Mg</td>
<td>0.79</td>
</tr>
<tr>
<td>$^{56}$Fe</td>
<td>0.65</td>
</tr>
<tr>
<td>$^{107}$Ag</td>
<td>0.55</td>
</tr>
<tr>
<td>$^{139}$Ba</td>
<td>0.51</td>
</tr>
<tr>
<td>$^{208}$Pb</td>
<td>0.46</td>
</tr>
<tr>
<td>$^{238}$U</td>
<td>0.44</td>
</tr>
</tbody>
</table>
The “Halo Nuclei”

$^{11}\text{Li}$

$^{208}\text{Pb}$
Nuclear Spin and Parity

• Nuclei have an intrinsic angular momentum. This angular momentum is referred to as the “spin.”

• For odd A nuclei, the spin J is half integer. (1/2, 3/2, etc.) For even A nuclei, the spin J is integer. (0, 1, 2, etc.). The magnitude of the spin is multiple of $\hbar$. 
Parity (the symmetry properties of the nuclear wave function)

\[ \Psi(r,s) = +\Psi(-r,-s) \]
\[ \pi = + \]
\[ \Psi(r,s) = -\Psi(-r,-s) \]
\[ \pi = - \]

For a central potential, \( V=V(r) \)

\[ \pi = (-1)^\ell \]
Nomenclature

\((J, \pi) = \frac{3}{2}^+\)

\((J, \pi) = 7^-\)
Nuclear Shapes

• Some nuclei are spherical. Many are not.
• Instead they can be football shaped or frisbee shaped.
Prolate football

Sphere baseball

Oblate doorknob
Electric and magnetic moments

- Electric moments are measures of the distribution of electric charge. Magnetic moments measure the distribution of electric currents.
Electric Quadrupole Moments

*Charge at point* = $\rho d\tau = \rho (r^2 dr \sin \theta d\theta d\phi)$

*Potential at* $P = d\Phi$

$$d\Phi = \frac{\rho d\tau}{\delta} = \frac{\rho d\tau}{(D^2 + r^2 - 2Dr \cos \theta)^{1/2}}$$

$$P_2(\cos \theta) = \frac{3}{2} \cos^2 \theta - \frac{1}{2}$$

$$P_1(\cos \theta) = \cos \theta$$

$$d\Phi = \frac{\rho d\tau}{D} \left[ 1 + \frac{r}{D} P_1(\cos \theta) + \left( \frac{r}{D} \right)^2 P_2(\cos \theta) + \cdots \right]$$

$$V = \frac{1}{D} \left[ \int_{\text{volume}} \rho d\tau \right] + \frac{1}{D^2} \left[ \int_{\text{volume}} \rho r \cos \theta d\tau \right] + \frac{1}{D^3} \left[ \int_{\text{volume}} \rho r^2 \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) d\tau + \cdots \right]$$
\[ Q = \iiint r^2 \rho(r) \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) r^2 \sin \theta \, dr \, d\theta \, d\phi \]

\[ R^2 = \frac{1}{2} (a^2 + c^2) = r_0 A^{1/3} \]

\[ Q = \frac{2}{5} Ze (a^2 - c^2) \]
For $^{177}$Hf, $Q = +3.0$ e-barns. Calculate the ratio of the semi-major to semi-minor axes of this prolate nucleus.

\[
\frac{Q}{e} = \frac{2}{5} \frac{Z}{Z} (a^2 - c^2)
\]

\[
a^2 - c^2 = \frac{Q}{e} \frac{2/5 * Z}{2/5(72)} = +3.0 \times 10^{-24} = 1.042 \times 10^{-25}
\]

\[
R^2 = \frac{1}{4}(a^2 + c^2) = \left( r_0 A^{1/3} \right)^2
\]

\[
a^2 + c^2 = 2\left( r_0 A^{1/3} \right)^2 = 2(1.2 \times 10^{-13} * 177^{1/3})^2 = 9.079 \times 10^{-25}
\]

\[
2a^2 = 1.012 \times 10^{-24} \quad a = 7.11 \times 10^{-13} \text{ cm} \quad c = 6.34 \times 10^{-13} \text{ cm}
\]

\[
a/c = 1.12
\]
Systematics of Electric Quadrupole Moments

Mostly prolate (\(Q>0\)) heavy nuclei

\[ \frac{Q}{Z R^2} \propto \frac{\Delta R}{R} \]

\(Q(167\text{Er}) = 30R^2\)

Odd-N
Odd-Z

Prolate

\(Q>0\): e.g., hole in spherical core → pattern not obvious. If such nuclei exist, weak effect of hole for \(Q\)

Tightly bound nuclei are spherical:
"Magic" \(N\) or \(Z\) = 8, 20, 28, 50, 82, 126, ...

\(Q<0\): e.g., extra particle around spherical core. Pattern recognizable

\[ 17\text{O}, 63\text{Cu}, 123\text{Sb}, 209\text{Bi} \]

\[ 27\text{Al}, 55\text{Mn}, 115\text{In}, 176\text{Lu}, 167\text{Er} \]
Q₀ Systematics

Ground-state quadrupole deformation

$Q₀$ large between magic N, Z numbers

$Q₀=0$ close to magic numbers

FRDM (1992)
Magnetic Moments
Classical Analogy

\[ |\mu| = iA = \left( \frac{ev}{2\pi r} \right) (\pi r^2) = \frac{evr}{2} \]

\[ \ell = mvr \]

\[ |\mu| = \frac{evr}{2} \cdot \frac{m}{m} = \frac{e\ell}{2m} \]

\[ \text{gyromagnetic ratio} = \gamma = \frac{|\mu|}{\ell} = \frac{e}{2m} \]
Quantum mechanical view

\[ \ell = m_\ell \hbar \]

\[ |\mu| = \frac{e}{2m} \ell = \frac{e}{2m} m_\ell \hbar = m_\ell \mu_B \]

\[ \mu_B = 5.78 \times 10^{-5} \frac{eV}{tesla} = 9.27 \times 10^{-21} \frac{erg}{gauss} \]

\[ \mu_B = Bohr \ magneton \]

\[ |\mu_{\text{proton}}| = m_\ell \mu_N \]

\[ \mu_N = \text{nuclear \ magneton} = 3.15 \times 10^{-8} \frac{eV}{tesla} = 5.50 \times 10^{-24} \frac{erg}{gauss} \]
Gyromagnetic ratios

\[ \gamma = \frac{e}{2m_p} \quad g = \frac{g\mu_N}{\hbar} \]

\( g \equiv g \text{ factor} \)

\[ \mu_\ell = g_\ell m_\ell \mu_N \]
\[ g_\ell(n) = 0 \]
\[ g_\ell(p) = 1 \]

\[ \mu_s = g_s m_s \mu_N \]
\[ g_s = 2.0023 \]

Find

\[ g_s(p) = 5.5856912 \]
\[ g_s(n) = -3.8260837 \]
Values of nuclear magnetic moments

- Value of the n, p moments—Chap. 5
- Value of the isotopic moments—Chap. 6