# Semi-Markov Cascade Representations of Local Solutions to 3d-Incompressible Navier-Stokes*†

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March 9, 2004

**Abstract**

A probabilistic approach introduced by LeJan and Sznitman (1997) permits derivation of weak solutions to 3-d incompressible Navier-Stokes equations whose Fourier transform may be represented by an expected value of a stochastic cascade. This approach was extended in Bhattacharya et al (2003) by methods which would yield unique global solutions by a stochastic representation under "small initial data conditions". A connection to iterative contraction maps on appropriate function space was also provided which would also yield local existence and uniqueness under "short time" constraints, but without stochastic cascade representations. In the present paper the authors (i.) Provide a stochastic cascade representation for local solutions, and (ii.) Provide time-asymptotics for global solutions from the stochastic representation.

*AMS 1991 subject classifications. Primary 35Q30,76D05; Secondary 60J80,76M3S. Key words and phrases: semi-Markov multiplicative stochastic cascade, branching random walk, incompressible Navier-Stokes, local solutions, time-asymptotics

†This research was partially supported by Focussed Research Group collaborative awards DMS-0073958, DMS-0073865 to Oregon State University and Indiana University/University of Arizona by the National Science Foundation.

# 1 Introduction & Preliminaries

A probabilistic approach was introduced by LeJan and Sznitman (1997) to obtain weak solutions to 3-d incompressible Navier-Stokes (NS) equations governing fluid velocities $u(x,t)$ given by

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u - \nabla p + g, \quad \nabla \cdot u = 0. \tag{1}$$

This approach is based on a representation of of the Fourier transform of $u(x,t)$ by an expected value of a multiplicative stochastic cascade. In Bhattacharya et al (2003) two related approaches are developed to obtain global and local existence, uniqueness and regularity, including spatial analyticity, of solutions to 3-dimensional incompressible Navier-Stokes (NS) equations. One approach is probabilistic and involves the construction of a multiplicative cascade solution to a related stochastic recursion in Fourier space which generalizes that of LeJan and Sznitman (1997). The other approach is based on Picard iterations in a suitably identified function space, generalizing those considered by Cannone and Planchon (2000), Cannone and Meyer (1995).

The probabilistic approach is based upon an interpretation of the integral equation governing Fourier transformed velocities scaled by a multiplier $1/h$. This is achieved in terms of the expected values of multiplicative cascade solutions to stochastic recursions generated by certain multi-type branching random walks in Fourier space. The transitions in wave-number are of the form $\xi \to (\xi_1, \xi_2), \xi_1 + \xi_2 = \xi$, with a transition probability kernel $h(\xi_1)h(\xi_2)/h * h(\xi)$. This generalizes branching random walks in the sense of LeJan and Sznitman (1997) for $k = 3$ dimensions where $h(\xi) = |\xi|^{-2}, \xi \in W_h^{(3)} = \mathbf{R}^3 \backslash \{0\}$. The holding time at wave number $\xi$ is *exponentially distributed* which, coupled with non-explosion, facilitated proofs through the use of the *strong Markov property* in both LeJan and Sznitman (1997) and in Bhattacharya et al (2003). More specifically, one considers the equation $(\mathrm{FNS})_h$ defined by

$$\chi(\xi,t) = e^{-\nu t|\xi|^2}\chi_0(\xi) + \int_0^t \nu|\xi|^2 e^{-\nu|\xi|^2 s}\{\cdots\}ds, \qquad (\mathrm{FNS})_h \quad (2)$$

where

$$\{\cdots\} = \{\frac{1}{2}m(\xi)\int_{W_h^{(k)}\times W_h^{(k)}} \chi(\eta_1, t-s)\otimes_\xi \chi(\eta_2, t-s)H(\xi, d\eta_1 \times d\eta_2) + \frac{1}{2}\varphi(\xi, t-s)\} \tag{3}$$

with

$$m(\xi) = \frac{2h * h(\xi)}{\nu(2\pi)^{\frac{3}{2}}|\xi|h(\xi)}, \quad \chi(\xi,t) = \frac{\hat{u}(\xi,t)}{h(\xi)}, \quad \varphi(\xi,t) = \frac{2\hat{g}(\xi,t)}{\nu|\xi|^2 h(\xi)}, \tag{4}$$

and $H(\xi, d\eta_1 \times d\eta_2)$ is the transition probability kernel defined by

$$\int f(\eta_1,\eta_2)H(\xi, d\eta_1 \times d\eta_2) = \int_{W_h^{(k)}} f(\xi-\eta,\eta)\frac{h(\xi-\eta)h(\eta)}{h * h(\xi)}d\eta \tag{5}$$

for bounded, Borel measurable $f$. Also

$$w \otimes_\xi z = -i(e_\xi \cdot z)\pi_{\xi^\perp}w, \; e_\xi = \frac{\xi}{|\xi|}, \text{ and } \pi_{\xi^\perp}w = w - (e_\xi \cdot w)e_\xi. \tag{6}$$

So $\pi_{\xi^\perp}$ is the projection of $w$ onto the plane orthogonal to $\xi$. The parameter $\nu > 0$ is the viscosity parameter. Note the exceptional role of $\xi = 0$ is linked to the use of the wave number $\xi$ in defining the exponential waiting time distribution with mean $1/\nu|\xi|^2$.

Constructions of a particular class of the Fourier multipliers $h$, referred to as *majorizing kernels,* satisfying an inequality of the form

$$h * h(\xi) \le B|\xi|^\theta h(\xi), \quad \xi \in W_h^{(3)}, \tag{7}$$

lead to either conditions for unique global solutions (with exponent $\theta = 1$) and/or local solutions (with exponent $0 < \theta < 1$.) The choice of the constant $B = 1$ may be taken without loss of generality. Such majorizing kernels are said to be *standard*. In particular, under assumptions on the size of the initial data and forcings which guarantee the finiteness of the indicated expected value, the probabilistic approach gives a representation of the Fourier transform $\hat{u}(\xi,t)$ of the global solution of the evolution equation in the LeJan-Sznitman form of an expected value

$$\hat{u}(\xi,t) = h(\xi)E_\xi \mathbb{X}(\xi,t). \tag{8}$$

Here $\mathbb{X}$ is a random multiplicative functional of scalar values $m(\cdot)$ and Fourier transformed initial data and/or forcing (vector) values over the vertices of a multi-type branching random walk tree initiated in time $t$ from a single progenitor of type $\xi$. In general the scalar and vector valued factors are evaluated at the wave-number (type) of the respective vertices appearing in the tree, with the initial and forcing terms appearing at the end-nodes. The exponentially distributed holding times between branchings are determined from the principal part of the equation, while the branching probabilities depend on the lower order and forcing terms of the equation.

The second approach in Bhattacharya et al (2003) is purely analytic in which the Fourier multiplier $1/h$ is used to identify a Banach space norm for which iterations of the expected values may, under slightly more restrictive conditions, be interpreted as Picard iterates of successive approximations on a suitably identified function space defined via particular control of the Fourier transform by a majorizing Fourier multiplier, e.g. $u \in \mathcal{S}'$ such that $|\hat{u}(\xi,t)| \leq h(\xi)$. In particular, the Picard iteration may be expressed in terms of a (strict) contraction operator on such a space. More specifically let us define a Banach space with a norm that depends on a Fourier multiplier $1/h$ as the completion of

$$\mathcal{F}_{h,\gamma,T} = \{v \in \mathcal{S}' : \hat{v}(\xi,t) = 0, \xi \in W_h^{(k)}, \|v\|_{\mathcal{F}_{h,\gamma,T}} = \sup_{\substack{\xi \in W_h^{(k)} \\ 0 \leq t < T}} \frac{|\hat{v}(\xi,t)|}{e^{-\gamma\sqrt{t}|\xi|}h(\xi)} < \infty\}, \tag{9}$$

where $\gamma \in \{0,1\}$ serves to conveniently index two different norms one may consider. Here $\mathcal{S}'$ is the space of tempered distributions on $\mathbf{R}^k$. Also, implicit to the definition of the Banach space $\mathcal{F}_{h,\gamma,T}$ is the requirement that tempered distributions belonging to this space have Fourier transforms which are functions. In the case $h(\xi) = |\xi|^{-2}$, $\mathcal{F}_{h,0,T}$ is the Besov type space introduced by Cannone and Planchon (2000). We will refer to such spaces $\mathcal{F}_{h,\gamma,T}$ as *majorizing spaces* in the case when $h$ is a majorizing kernel. The spaces $\mathcal{F}_{h,1,T}$ generalize those introduced by Lemarié-Rieusset (2000) to obtain conditions for spatial analyticity of solutions found by LeJan and Sznitman (1997).

**Remark 1.1** In order to restrict the solutions to correspond to (real) vector-valued incompressible flows, one may simply replace the Banach space $\mathcal{F}_{h,\gamma,T}$ by the closed subset

$$\mathcal{G}_{h,\gamma,T} = \{v \in \mathcal{F}_{h,\gamma,T} : \xi \cdot \hat{v}(\xi,t) = 0, \hat{v}(-\xi,t) = \overline{\hat{v}(\xi,t)}, \xi \in W_h^{(k)}, 0 \leq t \leq T\}. \tag{10}$$

**Remark 1.2** It may be noted that a function space for Picard iteration in physical (as opposed to Fourier) space-time was identified by Kato (1984) in efforts to obtain existence and uniqueness for Navier-Stokes equations. Kato's function space was expressed in terms of energy estimates.

The framework developed in Bhattacharya (2003) is also more generally applicable to diverse classes of evolution equations, including certain linear parabolic and fractional diffusion equations, semilinear reaction-diffusions, and some quasilinear equations such as incompressible Navier-Stokes equations in dimension $k \geq 2$, as well as one-dimensional Burgers' equation, and the generalized fractional Navier-Stokes and the fractional Burgers equation of the type considered by Biler, Funaki, and Woyczynski (1998). Even in the case of linear parabolic equations, where the branching is unary, for example a dual Feynman-Kac formula under the *complex measure condition* on coefficients is given by Itô (1965). In particular this approach makes Itô's complex measure condition completely natural from a probabilistic point of view; see Chen et al (2003), Kolokoltsov (2002), and references therein. Apart from the investigation of conditions for existence and uniqueness of solutions, the stochastic representation provides interesting new possibilities for numerical Monte Carlo simulations, e.g. see Ramirez (2004). Additionally the analysis of certain aspects of the behavior of solutions is made quite simple by the stochastic representation, e.g. time-asymptotics for global solutions. Results of this latter type will be included in the final section of the present paper. A focus of the present paper is the extension of the stochastic representation to local solutions of the 3d incompressible Navier-Stokes.

Local existence and uniqueness were obtained in Bhattacharya et al (2003) of the type illustrated by the following theorem.

**Theorem 1.1** *Let $h(\xi)$ be a standard majorizing kernel with exponent $\theta < 1$. Fix $0 < T \leq +\infty, \gamma \in \{0,1\}$. Assume $\|e^{\nu t\Delta}u_0(x)\| \in \mathcal{F}_{h,\gamma,T}$ and for some $1 \leq \beta \leq 2$, $(-\Delta)^{-\frac{\beta}{2}}g(x,t) \in \mathcal{F}_{h,\gamma,T}$ Then there is a $0 < T_* \leq T$ for which one has a unique solution $u \in \mathcal{F}_{h,\gamma,T_*}$.*

**Remark** Note that if $h$ is a majorizing kernel of exponent $\theta \leq 1$ and $u(x,t) \in \mathcal{F}_{h,\gamma,T} \cap \mathcal{C}^1((0,T),\mathcal{S}')$ is such that $\hat{u}(\xi,t)$ is a solution of the (FNS), $u = \check{\hat{u}}$ is a mild solution of the Navier-Stokes. Indeed, the definition of majorizing kernel and of the functional spaces $\mathcal{F}_{h,\gamma,T}$ imply that the product of distributions in $\mathcal{F}_{h,\gamma,T}$ is itself a distribution. To see this, note that if $u$ and $v$ are elements of $\mathcal{F}_{h,\gamma,T}$ for a standard majorizing kernel $h$ of exponent $\theta$, $|\hat{u} * \hat{v}(\xi)| \leq Mh * h(\xi) \leq M|\xi|^\theta h(\xi)$, where $M = \|u\|_{\mathcal{F}_{h,\gamma,T}}\|v\|_{\mathcal{F}_{h,\gamma,T}}$. Using the definition of a majorizing kernel, it follows that $\hat{u} * \hat{v}(\xi)$ is locally integrable, so in particular $\widehat{B(u,u)}(\xi,t) = \hat{B}(\hat{u},\hat{u})(\xi,t)$ as needed. Consequently, working in these function spaces, a direct relation between solutions obtained using the stochastic representation and the solutions obtained using Picard iteration methods can be obtained.

The organization of this paper is as follows. In the next Section 2 we describe an extension of the stochastic recursion associated with integral equations of the general form (FNS) which accommodates non-exponentially distributed holding times; a detailed proof of the theorem required for this extension is given in a PhD thesis by Orum (2004). In the next section this is applied to obtain local solutions for short times by a stochastic cascade construction. Finally we conclude this paper with results on the time-asymptotic behavior of unique global solutions obtained from the stochastic cascade representation.

# 2 Semi-Markov Cascades and Local Representations

Let us consider integral equations of the general form

$$\chi(\xi,t) = a(\xi,t)p(\xi,[t,\infty)) + \int_0^t \{\cdots\}p(\xi,ds), \tag{11}$$

where

$$\{\cdots\} = \frac{1}{2}b(\xi,s)\int_{R^k\backslash\{0\}} \chi(\eta,t-s)\otimes_\xi \chi(\xi-\eta,t-s)q(\xi,d\eta)+\frac{1}{2}c(\xi,s,t-s), \tag{12}$$

and $A \to p(\xi,A), A \in \mathcal{B}([0,\infty))$, and $B \to q(\xi,B), B \in \mathcal{B}(R^k\backslash\{0\})$, are probability measures for each $\xi \in R^k\backslash\{0\}$.

The following examples include natural rescalings of standard equations that put them in the form (11). However they are cast in terms of more general rescaling which provides more flexibility. For fixed $\xi$, let $G(\xi,t), t \geq 0$, be an arbitrary probability distribution function defined by a continuous positive failure rate $\lambda(\xi,t) > 0, t \geq 0$, i.e. $\int_0^\infty \lambda(\xi,t)dt = \infty$, and

$$G(\xi,t) := 1 - \exp(-\int_0^t \lambda(\xi,s)ds), \quad t \geq 0. \tag{13}$$

In the following examples we assume only that $h : R^k\backslash\{0\} \to (0,\infty)$ is measurable and $h * h(\xi) < \infty$. We do *not* require that $h$ necessarily be a majorizing kernel with exponent $\theta$ for the general formulation, although such considerations will emerge in the subsequent analysis.

**Example 1.** *(Incompressible Navier-Stokes).* Take $k \geq 2$. Then (11) is obtained by Fourier transformation and orthogonal projection of a solution to the $k-$dimensional incompressible Navier-Stokes with

$$\chi(\xi, t) := \frac{\hat{u}(\xi, t)}{h(\xi)}, \qquad \varphi(\xi, t) := \frac{\hat{g}(\xi, t)}{h(\xi)}, \qquad \beta(\xi) := \frac{h * h(\xi)}{h(\xi)} \tag{14}$$

and

$$p(\xi, [t, \infty)) := 1 - G(\xi, t) = \exp(-\int_0^t \lambda(\xi, s) ds). \tag{15}$$

In particular $p(\xi, dt)$ has the pdf $G'(\xi, t) = \lambda(\xi, t) e^{-\int_0^t \lambda(\xi, s) ds} \mathbf{1}_{[0,\infty)}(t)$. Also, define

$$q(\xi, d\eta) := \frac{h(\eta) h(\xi - \eta)}{h * h(\xi)} d\eta. \tag{16}$$

Then

$$a(\xi, t) = m_0(\xi, t) \chi_0(\xi) = \exp\{-\int_0^t (\nu |\xi|^2 - \lambda(\xi, s)) ds\} \chi_0(\xi), \tag{17}$$

and

$$b(\xi, s) = \frac{2(2\pi)^{-\frac{k}{2}}}{\lambda(\xi, s)} |\xi| \beta(\xi) \exp\{-\int_0^s (\nu |\xi|^2 - \lambda(\xi, u)) du\}. \tag{18}$$

Finally,

$$c(\xi, s, t - s) = \frac{2}{\lambda(\xi, s)} \exp\{-\int_0^s (\nu |\xi|^2 - \lambda(\xi, u)) du\} \varphi(\xi, t - s). \tag{19}$$

The (vector) product operation is defined by

$$w \otimes_\xi z = -i(e_\xi \cdot z) \pi_{\xi^\perp} w, \;\; e_\xi = \frac{\xi}{|\xi|}, \text{ and } \pi_{\xi^\perp} w = w - (e_\xi \cdot w) e_\xi. \tag{20}$$

It may be noted that replacing the Laplacian by a *fractional Laplacian operator* can be accomodated by minor adjustments.

Assuming for example that $\hat{u}_0(\xi) \in L^1$, observe that one may take $h(\xi) = |\hat{u}_0(\xi)|$. In this case $\chi_0(\xi) \equiv 1$.

**Example 2.** *(Quasi-linear Burgers and Fractional Burgers Equations)* Burgers' equation is the scalar equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial u^2}{\partial^2 x} - u\frac{\partial u}{\partial x} + g, \tag{21}$$

with initial data $u(x, 0) = u_0(x), x \in R$. One may also include $k-$dimensional spatial variables $\mathbf{x}$, but the function $u(\mathbf{x}, t)$ is a one-dimensional scalar. This example is often regarded as the "one-dimensional" version of Example 1. However, the absence of a notion of incompressibility in one dimension is a major point of distinction. Since in one-dimension there will be no projection, each of the $k = 1$ versions of the expressions defining $\chi(\xi, t)$,$\varphi(\xi, t)$, $p(\xi, dt)$, $q(\xi, d\eta)$, $a(\xi, t)$, $\beta(\xi)$, and $c(\xi, s, t-s), s, t \geq 0, \xi \in R^1$ for Navier-Stokes will apply. Since $u\partial u/\partial x = \frac{1}{2}\partial u^2/\partial x$, the same is true for $b(\xi, t)$ up to the factor of 2 which cancels. Also, apart from the complex factor $-i \equiv -\sqrt{-1}$, the multiplication operator $\otimes_\xi$ is ordinary arithmetic multiplication of complex numbers. The modification required for fractional differentiation of the type considered by Biler et al (1998) is a minor modification from the Fourier perspective.

**Example 3.** *(Linear Diffusion and Pseudo-differential Equations)* If the nonlinear term $uu_x$ is omitted from Burgers equation then the scalar forced heat equation remains. In this case the stochastic recursion is birth-death in the sense that the multiplication is either terminated or joined by a single new factor at each random clock ring. The same holds true for the $k-$dimensional scalar equation

$$\frac{\partial u}{\partial t} = Lu - v(\mathbf{x})u + g, \quad u(\mathbf{x}, 0) = u_0(\mathbf{x}), \tag{22}$$

where the Fourier transform of $v$ is assumed to be a complex measure, and $L$ is a pseudo-differential operator with symbol $\gamma(\mathbf{x}, \xi)$. That is $\gamma : R^k \times R^k \to C$ is a measurable function such that $\xi \to \gamma(\mathbf{x}, \xi)$ is continuous and polynomial bounded, and for $u \in \mathcal{S}(R^k)$,

$$Lu(\mathbf{x}) := (2\pi)^{-\frac{k}{2}} \int_{R^k} e^{i\mathbf{x}\cdot\xi}\gamma(\mathbf{x}, \xi)\hat{u}(\xi)d\xi \tag{23}$$

To simplify the representation we assume that $\hat{v}(d\xi)$ is a probability measure. In the absence of this assumption one further decomposes $\hat{v}$ into real and imaginary parts, followed by a corresponding Hahn decomposition; see Chen et al(2003). Similarly, for simplicity assume $\gamma(\mathbf{x},\xi) \equiv \gamma(\xi)$. Basic examples are furnished by symbols for fractional derivatives. One may note that large classes of integro-differential equations are included in the generality of this formulation. Under these assumptions one has

$$\begin{aligned}
&p(\xi,[t,\infty)) := 1 - G(\xi,t) = \exp\Big\{-\int_0^\infty \lambda(\xi,s)ds\Big\}\\
&m_0(\xi,t) = \exp\Big\{\int_0^t (\gamma(\xi)+\lambda(\xi,s))ds\Big\}\\
&a(\xi,t) = m_0(\xi,t)\hat{u}_0(\xi), \quad b(\xi,s) = -\frac{2}{(2\pi)^{\frac{k}{2}}\lambda(\xi,s)}m_0(\xi,s)\\
&c(\xi,s,t-s) = \frac{2}{\lambda(\xi,s)}m_0(\xi,s)\hat{g}(\xi,t-s), \quad q(\xi,d\eta) = \hat{v}(d\eta)
\end{aligned} \tag{24}$$

Since there is no binary branching in this case, the operation $\otimes_\xi$ is unary and does not explicitly appear.

**Example 4.** *(Semi-linear Reaction-Diffusion)* Here let us consider the $k-$dimensional semi-linear scalar KPP equation

$$\frac{\partial u}{\partial t} = \nu\Delta u + u(u-1) + g, \quad u(\mathbf{x},0) = u_0(\mathbf{x}). \tag{25}$$

In this case the coefficients of the general equation are given by

$$\begin{aligned}
&p(\xi,[t,\infty)) := 1 - G(\xi,t) = \exp(-\int_0^\infty \lambda(\xi,s)ds)\\
&m_0(\xi,t) = \exp\{\int_0^t (\lambda(\xi,s) - \nu|\xi|^2 - 1)ds\}\\
&a(\xi,t) = m_0(\xi,t)\frac{\hat{u}_0(\xi)}{h(\xi)}, \quad b(\xi,s) = \frac{2}{(2\pi)^{\frac{k}{2}}\lambda(\xi,s)}m_0(\xi,s)\\
&c(\xi,s,t-s) = \frac{2\beta(\xi)}{\lambda(\xi,s)}m_0(\xi,s)\frac{\hat{g}(\xi,t-s)}{h(\xi)}, \; q(\xi,d\eta) = \frac{h(\xi)h(\xi-\eta)}{h*h(\xi)}d\eta).
\end{aligned} \tag{26}$$

Here, as before, $\beta(\xi) = h*h(\xi)/h(\xi)$. Finally, $w \otimes_\xi z = zw$ is ordinary multiplication of complex numbers. Once again the extension to pseudo-differential reaction-diffusion equations is straightforward.

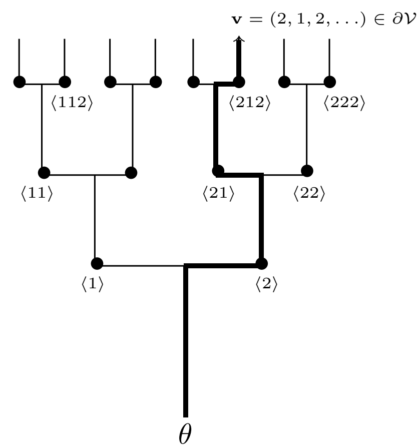


Figure 1: Full binary tree with index set $\mathcal{V}$ and boundary $\partial\mathcal{V}$. The path $\mathbf{v} = (2,1,2,\ldots) \in \partial\mathcal{V}$ is indicated in bold, with $\mathbf{v}|0 = \theta$, $\mathbf{v}|1 = \langle 2\rangle$, $\mathbf{v}|2 = \langle 21\rangle$, and $\mathbf{v}|3 = \langle 212\rangle$.

**Remark** In each of Examples 2-4 a stochastic representation is possible on the real space side as well on the Fourier side. To date the same cannot be said for Example 1.

The probabilistic interpretation of the general equation (11) is initiated by viewing it as a recursion of expected values over a stochastic cascade in which the holding times are distributed according to the probability distribution $p(\xi, dt)$. To construct this stochastic cascade requires some additional notation and terminology. The vertex set $\mathcal{V}$ of a complete binary tree rooted at $\theta$ may be coded as (see Figure 1)

$$\mathcal{V} = \cup_{j=0}^{\infty}\{1,2\}^j = \{\theta, <1>, <2>, <11>, \ldots\}, \tag{27}$$

where $\{1,2\}^0 = \{\theta\}$. Also let $\partial\mathcal{V} = \prod_{j=0}^{\infty}\{1,2\} = \{1,2\}^{\mathbf{N}}$.

A stochastic model consistent with (11) is obtained by consideration of a multitype branching random walk of nonzero Fourier wavenumbers $\xi$, thought of as particle *types*, as follows: A particle of type $\xi \neq 0$ initially at the root $\theta$ holds for a random length of time $S_\theta$ distributed according to $p(\xi, dt)$.

When this clock rings, a coin $\kappa_\theta$ is tossed and either with probability $\frac{1}{2}$ the event $[\kappa_\theta = 0]$ occurs and the particle is terminated, or with probability $\frac{1}{2}$ one has $[\kappa_\theta = 1]$, the clocks are re-set and the particle is replaced by two offspring particles of types $\eta, \xi-\eta$ selected according to the probability kernel $q(\xi, d\eta)$. This process is repeated independently for the particle types $\eta$ and $\xi - \eta$ rooted at the vertices $< 1 >, < 2 >$, respectively.

A more precise description of the stochastic model requires more notation. For $\mathbf{v} = (v_1, v_2, \ldots, v_k) \in \mathcal{V}$, let $|\mathbf{v}| = k,\ |\theta| = 0$. For $\mathbf{v} = (v_1, v_2, \ldots) \in \partial\mathcal{V}$, and $j = 0, 1, 2, \ldots$ let $\mathbf{v}|j = (v_1, \ldots v_j),\ \mathbf{v}|0 = \theta$. That is, for $\mathbf{v} \in \partial\mathcal{V},\ \mathbf{v}|0, \mathbf{v}|1, \mathbf{v}|2, \ldots$ may be viewed as a *path* through vertices of the tree starting from the root $\mathbf{v}|0 = \theta$. For $\mathbf{u}, \mathbf{v} \in \partial\mathcal{V}$, let $|\mathbf{u} \wedge \mathbf{v}| = \inf\{m \geq 1 : \mathbf{u}|m \neq \mathbf{v}|m\}$.

The following requirements provide the defining properties of the underlying stochastic model. The model depends on the initial frequency (wave number) $\xi$. For fixed $\xi \neq 0$ let $\{(\xi_v, \kappa_v) : \mathbf{v} \in \mathcal{V}\}$ be the tree-indexed (discrete parameter) Markov process starting at $(\xi_\theta, \kappa_\theta)$ with $\xi_\theta = \xi,\ \kappa_\theta \in \{0, 1\}$, taking values in the state space $(R^3\backslash\{0\}) \times \{0, 1\}$, and defined on a probability space $(\Omega, \mathcal{F}, P_\xi)$ by the following properties:

1. $P_\xi(\xi_\theta \in B,\ \kappa_\theta = \kappa) = \frac{1}{2}\delta_\xi(B),\ B \in \mathcal{B}_h,\ \kappa \in \{0, 1\}$.

2. For any fixed $\mathbf{v} \in \partial\mathcal{V}$, the sequence $(\xi_{v|0},\ \kappa_{v|0}),\ (\xi_{v|1},\ \kappa_{v|1}),\ (\xi_{v|2},\ \kappa_{v|2}), \ldots$ is a Markov chain with transition probabilities
$$\begin{aligned} P_\xi(\xi_{v|n+1} \in B,\ \kappa_{v|n+1} = \kappa | \sigma(\{(\xi_u, \kappa_u) : |u| \leq n\})) \\ = \tfrac{1}{2} q(\xi, B) \end{aligned} \tag{28}$$
for $B \in \mathcal{B}(R^3\backslash\{0\}), \kappa \in \{0, 1\}$.

3. For any $\mathbf{u}, \mathbf{v}, \in \partial\mathcal{V},\ \{(\xi_{\mathbf{u}|m}, \kappa_{\mathbf{u}|m})\}_{m=0}^\infty$ and $\{(\xi_{v|m}, \kappa_{v|m})\}_{m=0}^\infty$ are conditionally independent given $\sigma(\{(\xi_w, \kappa_w) : |\mathbf{w}| \leq |\mathbf{u} \wedge \mathbf{v}|\})$.

4. For $\mathbf{v} \in \mathcal{V},\ \xi_{v1} + \xi_{v2} = \xi_v \quad P_\xi - a.s.$, where $vj = (v_1 \ldots v_n)j := (v_1 \cdots v_n, j),\ j = 1, 2, \ldots$ is the concatenation operation.

5. Let $\{S_v : v \in \mathcal{V}\}$ be a collection of non-negative random variables such that for each $m \geq 1$, conditionally given $\xi_\theta = \xi$ and $\xi_\theta^+ := \{\xi_v : v \in \mathcal{V}, |v| \geq 1\}$, the random variables $S_v, v \in \mathcal{V}$ are independent with respective (conditional) marginal distributions $p(\xi_v, ds)$.

Our objective now is to use the stochastic branching model to recursively define a random functional related to (11) through its expected value. By a backward recursion one may define a (non-random) function $f(z, z^+, s, s^+, \kappa, \kappa^+, t)$ where $z \in R^k\backslash\{0\}, z^+ \in (R^k\backslash\{0\})^{\mathcal{V}^+}, s \in [0, \infty), s^+ \in [0, \infty)^{\mathcal{V}^+}$, and $\kappa \in \{1, 2\}, \kappa^+ \in \{1, 2\}^{\mathcal{V}^+}$, where

$$\mathcal{V}^+ := \{v \in \mathcal{V} : |v| \geq 1\}, \tag{29}$$

such that

$$\begin{aligned} & f(z, z^+, s, s^+, \kappa, \kappa^+, t) = a(z, t)\mathbf{1}_{[t,\infty)}(s) + \mathbf{1}_{[0,t)}(s)\mathbf{1}_{\{1\}}(\kappa)b(z, t-s) \\ \cdot \quad & f(z_1, z_1^+, s_1, s_1^+, \kappa_1, \kappa_1^+, t-s) \otimes_z f(z_2, z_2^+, s_2, s_2^+, \kappa_2, \kappa_2^+, t-s) \\ + \quad & \mathbf{1}_{[0,t)}(s)\mathbf{1}_{\{0\}}(\kappa)c(z, s, t-s), \end{aligned} \tag{30}$$

where for $x \in S^{\mathcal{V}}$ we define

$$x^+ \equiv x_\theta^+ := (x_1, x_2, x_{11}, x_{12}, x_{21}, \ldots) \in S^{\mathcal{V}^+} \tag{31}$$

$$x_v^+ := (x_{v1}, x_{v2}, x_{v11}, x_{v12}, x_{v21}, x_{v22}, \ldots). \tag{32}$$

In the event that $\kappa_v = 1$ for all $v$ the recursion may not terminate. In this case one simply defines $f(z, z^+, s, s^+, \kappa, \kappa^+, t) \equiv 0$). Otherwise the backward recursion is sure to terminate and $f$ is well-defined.

For this given function $f$ let us now define a random functional of the random fields $\{\xi_v : v \in \mathcal{V}\}, \{\kappa_v : v \in \mathcal{V}\}$, and $\{S_v : v \in \mathcal{V}\}$ defined on $(\Omega, \mathcal{F}, P)$ by the composition

$$\mathbb{X}(\theta, t)(\omega) := f(\xi_\theta(\omega), \xi_\theta^+(\omega), S_\theta(\omega), S_\theta^+(\omega), \kappa_\theta(\omega), \kappa_\theta^+(\omega), t), \omega \in \Omega. \tag{33}$$

A careful formulation and details of a proof is given in a PhD thesis by Orum (2004) which yields the following:

**Theorem 2.1** *If $E|\mathbb{X}(\theta, t)| < \infty$ then a solution of (11) is given by*

$$\chi(\xi, t) := E_{\xi_\theta = \xi}\mathbb{X}(\theta, t).$$

We will conclude this section with an application to local existence theory for Navier-Stokes via a semi-Markov cascade representation. Taking a constant failure rate $\lambda(\xi, t) = \delta > 0$, this includes a special case obtained by Orum (2004) of local existence having exponentially distributed holding times.

**Theorem 2.2** *Assume that there is a* $0 < T_* \le \infty$ *such that for all* $\xi \in R^k \backslash \{0\}$, $\max\{|a(\xi,t)|, |b(\xi,t)|, \sup_{0\le s\le t}|c(\xi,s,t-s)|\} \le 1$ *for* $0 < t < T_*$. *Then* $\chi(\xi,t) = E_{\xi_\theta=\xi}\rlap{\chi}\chi(\theta,t)$ *solves (11) for* $0 \le t \le T_*$ *with* $|\chi(\xi,t)| \le 1$.

The following corollary is obtained by considering local solutions to (FNS) in spaces $\mathcal{F}_{h,0,T}$ where $h$ is an exponent $\theta$ majorizing kernel for some $0 \le \theta < 1$. For this application $G$ is a gamma probability distribution with shape parameter $\frac{1+\theta}{2}$ and scale parameter $p\nu|\xi|^2$ for fixed $0 < p < 1$.

**Corollary 2.1** *Let* $h$ *be a majorizing kernel of exponent* $0 \le \theta < 1$. *Assume* $u_0 \in \mathcal{F}_{h,0,0}$, *and* $(-\Delta)^{-\frac{1+\theta}{2}} g \in \mathcal{F}_{h,0,T}$ *for some* $T$. *Then there is a* $T_* \le T$ *and a unique solution* $u \in \mathcal{F}_{h,0,T_*}$ *to (FNS).*

**Proof** Since $u_0 \in \mathcal{F}_{h,0,0}$ for a majorizing kernel $h$ with exponent $\theta \in [0,1)$, by re-scaling $h$ we can assume that $h$ has constant $B$ and

$$|u_0|_{\mathcal{F}_{h,0,0}} \le \frac{1}{\Gamma(\frac{1+\theta}{2})} \inf_{\alpha>0} e^\alpha \int_{s=p\alpha}^\infty s^{\frac{\theta-1}{2}} e^{-s} ds \quad \text{for some} \quad p \in (0,1).$$

For $t \le T$, set

$$R_\theta(t) := \sup_{0\le s\le t, \xi\in W_h^{(k)}} |\xi|^{-(1+\theta)} \frac{|\hat{g}(\xi,s)|}{h(\xi)}$$

and $M_p(\theta) = 2(2\pi)^{-\frac{k}{2}}\Gamma(\frac{1+\theta}{2})(p\nu)^{-\frac{1+\theta}{2}}$. Take

$$T_\theta = \min(T, (BM_p(\theta))^{-\frac{2}{1-\theta}}).$$

and define

$$T_* = \min(T_\theta, ((2\pi)^{\frac{k}{2}} M_p(\theta) R_\theta(T_\theta))^{-\frac{2}{1-\theta}}).$$

Take

$$\frac{\partial G}{\partial t}(\xi,t) = \frac{(p\nu|\xi|^2)^{\frac{1+\theta}{2}} t^{\frac{\theta-1}{2}} e^{-p\nu|\xi|^2 t}}{\Gamma(\frac{1+\theta}{2})} \mathbf{1}_{[0,\infty)}(t),$$

one has

$$m_0(\xi,t) = \frac{\Gamma(\frac{1+\theta}{2})e^{-\nu|\xi|^2 t}}{\int_{s=t}^{\infty}(p\nu|\xi|^2)^{\frac{1+\theta}{2}} s^{\frac{\theta-1}{2}} e^{-p\nu|\xi|^2 s}ds}$$
$$\leq \Gamma(\frac{1+\theta}{2}) \sup_{\alpha>0} \frac{e^{-\alpha}}{\int_{s=p\alpha}^{\infty} s^{\frac{\theta-1}{2}} e^{-s}ds}. \tag{34}$$

So

$$|a(\xi,t)| = |\chi_0(\xi)| \cdot |m_0(\xi,t)| \leq 1.$$

Recall that by definition of a majorizing kernel of exponent $\theta$ and constant $B$ one has $\beta(\xi) = \frac{h*h(\xi)}{h(\xi)} \leq B|\xi|^\theta$, Thus, for $t \leq T_\theta$,

$$b(\xi,t) = 2(2\pi)^{-\frac{k}{2}}|\xi|\beta(\xi)\Gamma(\frac{1+\theta}{2})(p\nu|\xi|^2)^{-\frac{1+\theta}{2}} t^{\frac{1-\theta}{2}} e^{-(1-p)\nu|\xi|^2 t}$$
$$\leq 2(2\pi)^{-\frac{k}{2}} B\Gamma(\frac{1+\theta}{2})(p\nu)^{-\frac{1+\theta}{2}} T_\theta^{\frac{1-\theta}{2}}$$
$$= BM_p(\theta)T_\theta^{\frac{1-\theta}{2}} \leq 1. \tag{35}$$

Finally for $t \leq T_*$,

$$|c(\xi,s,t-s)| = 2\Gamma(\frac{1+\theta}{2})\frac{(p\nu|\xi|^2)^{-\frac{1+\theta}{2}} s^{\frac{1-\theta}{2}} e^{-(1-p)\nu|\xi|^2 s}|\hat{g}(\xi,t-s)|}{h(\xi)}$$
$$\leq (2\pi)^{\frac{k}{2}} M_p(\theta) t^{\frac{1-\theta}{2}} R_\theta(t)$$
$$\leq (2\pi)^{\frac{k}{2}} M_p(\theta) T_\theta^{\frac{1-\theta}{2}} R_\theta(T_\theta) \leq 1. \tag{36}$$

Thus the Corollary follows. ■

**Example** To illustrate the Corollary in the context of a specific example take $k = 3$ and let $\theta \in [0,1)$ and $r \geq 4 - 3\theta$. The function

$$h_{\theta,r}(\xi) = |\xi|^{-\theta}|(1+|\xi|)^{-r}$$

may be checked to be a majorizing kernel with exponent $\theta$; in fact constructed as a log-convex combination of majorizing kernels, see Bhattacharya et al (2003). For initial data and forcings satisfying Fourier transform decay of the form

$$\sup_\xi |\xi|^\theta(1+|\xi|)^r|u_0(\xi)| < \infty$$

and for some $t > 0$,

$$\sup_{\xi, 0 \le s \le t} |\hat{g}(\xi, s)||\xi|^{2\theta-1}(1+|\xi|)^r < \infty$$

one obtains a local solution to (FNS) in $\mathcal{F}_{h_{\theta,r},0,T_*}$.

# 3 Time-Asymptotic Steady State Solutions

In this section we compute time-asymptotics under supplemental conditions for the existence of a unique global solution to 3-d incompressible Navier-Stokes equations. For this we begin with the following theorem from Bhattacharya et al (2003).

**Theorem 3.1** *Let $h(\xi)$ be a standard majorizing kernel with exponent $\theta = 1$. Fix $0 < T \le +\infty$. Suppose that $\|u_0\|_{\mathcal{F}_{h,0,T}} \le (\sqrt{2\pi})^3\nu/2$ and $\|(-\Delta)^{-1}g\|_{\mathcal{F}_{h,0,T}} \le (\sqrt{2\pi})^3\nu^2/4$. Then there is a unique solution $u$ in the ball $B_0(0,R)$ centered at $0$ of radius $R = (\sqrt{2\pi})^3\nu/2$ in the space $\mathcal{F}_{h,0,T}$. Moreover the Fourier transform of the solution is given by $\hat{u}(\xi,t) = h(\xi)E_\xi \chi(\tau(\xi,t)), \xi \in W_h^{(3)}$ and $t \le T$.*

**Corollary 3.1** *Under the conditions of Theorem 3.1 with $T = \infty$, suppose further that $\lim_{t\to\infty} \hat{g}(\xi,t) = \hat{g}_\infty(\xi)$ exists for each $\xi \neq 0$. Then*

$$\hat{u}_\infty(\xi) := \lim_{t\to\infty} \hat{u}(\xi,t)$$

*exists and satisfies the steady state Navier-Stokes*

$$\hat{u}_\infty(\xi) = \int_0^\infty e^{-\nu|\xi|^2 s}\{|\xi|(2\pi)^{-\frac{3}{2}} \int_{\mathbf{R}^3} \hat{u}_\infty(\eta) \otimes_\xi \hat{u}_\infty(\xi-\eta)d\eta + \hat{g}_\infty(\xi)\}ds. \quad (\text{FNS})_\infty$$

*Proof* Observe that the underlying discrete parameter binary branching is critical. Thus $\lim_{t\to\infty} \chi(\theta,t)$ exists a.s. as a finite random product. Moreover, under the conditions of Theorem 3.1 one has for each $t \ge 0$, with probability one $|\chi(\theta,t)| \le 1$. Thus, by Lebesgue's Dominated Convergence Theorem

$$\chi_\infty(\xi) := \lim_{t\to\infty} \chi(\xi,t)$$

exists for each $\xi$. Now again apply Dominated Convergence to $(FNS)_h$ to obtain

$$\chi_\infty(\xi) = \int_0^\infty \nu|\xi|^2 e^{-\nu|\xi|^2 s}\{\frac{1}{2}m(\xi)\int_{\mathbf{R}^3}\chi_\infty(\eta)\otimes_\xi\chi_\infty(\xi-\eta)q_\xi(d\eta)+\frac{1}{2}\varphi_\infty(\xi)\}ds. \tag{37}$$

Multiplication through by $h(\xi)$ proves the assertion for $\hat{u}_\infty(\xi) = h(\xi)\chi_\infty(\xi)$ and $\varphi_\infty(\xi) = \frac{2\hat{g}_\infty(\xi)}{\nu|\xi|^2 h(\xi)}$. ■

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