MTH 306H MIDTERM I REVIEW PROBLEMS

0. Review the homework and quizzes!

1. a) Find the real and imaginary parts of the complex number $z = \dfrac{2-i}{1+i}$.

   b) Describe the set of points in the complex plane satisfying the inequality $|z+i| \leq 1$.

   c) Find the domain of the function $f(z) = \dfrac{z+1}{\mathrm{Re}(z)+\mathrm{Im}(z)}$.

2. Let $f(x) = \cos x$.

   a) Find the third degree Taylor polynomial $P_3(x)$ of $f(x)$ centered at $a = 0$.

   b) Write down an expression for the third degree remainder term $R_3(x)$ of $f(x)$ for $a = 0$, and show that if $|x| < 0.1$ then $|R_3(x)|$ is less than $5 \cdot 10^{-6}$.

3. a) What is the maximum error possible in using the approximation $\sin x \approx x - x^3/3! + x^5/5!$ if $0 \leq x \leq 0.3$?

   b) For what values of x is this approximation accurate to within 0.00005?

4. Find the nth degree Taylor polynomial of $f(x) = \sin x$ centered at $a = \pi/4$. Find an expression for the remainder $R_n(x)$ and show that $\lim_{n\to\infty} R_n(x) = 0$ for all x. What do you conclude?

5. Find the eigenvalues and eigenvectors of the matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 3 & -2 \end{pmatrix}$.

6. Are the vectors $\mathbf{u} =< 1, 2, -2 >$, $\mathbf{v} =< 3, 1, 0 >$, $\mathbf{w} = < 2, -1, 1 >$ linearly independent?

7. Solve the system of equations

$$\begin{cases} 2x + 3y = -6 \\ -x + 2y = 1 \end{cases}$$

   by the Gaussian elimination using an augmented matrix.

8. Use an augmented matrix and row reduction to find the general solution of the system

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= -3 \end{aligned}$$

9. Let $\mathbb{A}$ be the 2×2 matrix $\mathbb{A} = \begin{pmatrix} 7 & 6 \\ 6 & -2 \end{pmatrix}$.

   a) Find the eigenvalues of $\mathbb{A}$.
   b) Find all eigenvectors of $\mathbb{A}$.
   c) Is the matrix $\mathbb{A}$ is diagonalizable. Why or why not?

10. Let $\mathbb{A} = \begin{pmatrix} 2 & -5 & -1 \\ -6 & 0 & 10 \\ 3 & 10 & h \end{pmatrix}$, where h is a real number.

    a) Find $\det \mathbb{A}$.
    b) For what values of h is the matrix $\mathbb{A}$ invertible?

11. a) Find a numerical series whose sum is $\int_0^2 x^3 e^{-x^2}\, dx$.
    b) How many terms of the series in part a do we need to add in order to approximate the integral $\int_0^2 x^3 e^{-x^2}\, dx$ within an error of 10^{-3}?

12. Suppose that matrices $\mathbb{C}$, $\mathbb{D}$ have the inverse matrices $\mathbb{C}^{-1} = \begin{pmatrix} 7 & 6 \\ 6 & -2 \end{pmatrix}$, $\mathbb{D}^{-1} = \begin{pmatrix} -1 & 6 \\ 2 & -2 \end{pmatrix}$. Find $(\mathbb{D}\mathbb{C})^{-1}$.

13. Let $\mathbf{F}$ be the vector space of all functions from $\mathbf{R}$ into $\mathbf{R}$. Let $T\colon \mathbf{F} \to \mathbf{R}^3$ be the mapping $T(f) = \begin{bmatrix} f(-1) \\ f(0) \\ f(1) \end{bmatrix}$. Is T a linear transformation? Justify your answer.

14. Let $\mathbb{A}$ be an $n \times n$ matrix and suppose that there is an $n \times n$ matrix $\mathbb{C}$ so that $\mathbb{C}\mathbb{A} = \mathbb{I}$, where $\mathbb{I}$ is the identity matrix.

    a) Show that the matrix equation $\mathbb{A}\mathbf{x} = \mathbf{0}$ only has the trivial solution.
    b) Next explain why the result of part a implies that $\mathbb{A}$ can be reduced to the identity matrix $\mathbb{I}$ by a repeated application of row operations.
    c) Finally use the result of part b to show that $\mathbb{A}$ is an invertible matrix.

15. a) Find the standard matrix of the linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ that first performs a vertical shear mapping $\mathbf{e}_1$ into $\mathbf{e}_1 - 3\mathbf{e}_2$ (leaving $\mathbf{e}_2$ unchanged) and then reflects the result about the x_2-axis.
    b) Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation such that $T\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $T\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$. Find $T(2\mathbf{u} - \mathbf{v})$.
    c) Give a geometric description of the linear tranformation associated with the matrix

$$\mathbb{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

16. a) Let T_A, T_B be linear transformations with the standard matrices $\mathbb{A} = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix}$, $\mathbb{B} = \begin{bmatrix} 0 & 3 \\ 2 & 1 \\ 5 & 1 \end{bmatrix}$. Find the matrix associated with the composition $T_A \circ T_B$.
    b) Let $\mathbb{A}$ be the 1×3 matrix $\mathbb{A} = [\,2 \quad 3 \quad -1\,]$. Compute the products $\mathbb{A}\mathbb{A}^T$ and $\mathbb{A}^T\mathbb{A}$.

17. a) Let $\mathbb{A}$ be an invertible $n \times n$ matrix. Prove that the linear transformation T_A associated with $\mathbb{A}$ must be one-to-one and onto.
    b) Find the inverse matrix of $\mathbb{A} = \begin{pmatrix} 2 & 6 \\ 3 & 1 \end{pmatrix}$ by row reducing the augmented matrix $[\mathbb{A}\ \mathbb{I}]$, where $\mathbb{I}$ is the identity matrix.

18. a) Detemine whether the functions $f_0(x) = 1$, $f_1(x) = \exp(x)$, $f_2(x) = \exp(2x)$, $f_3(x) = \exp(3x)$ are linearly independent or dependent.
    b) Determine whether the vectors

$$\mathbf{v}_1 =< 1, 1, 2 >, \quad \mathbf{v}_2 =< 2, -1, 1 >, \quad \mathbf{v}_3 =< -4, 5, 1 >$$

are linearly independent or dependent. If they are dependent, exhibit a non-trivial relation amongst the three vectors.

19. Let $\mathbb{A}$ be a real symmetric matrix.
    a) Show that the eigenvalues of $\mathbb{A}$ must be real. (Hint: Consider the expression $(\mathbb{A}\mathbf{v}) \cdot \overline{\mathbf{v}}$, where the bar stands for complex conjugation.)
    b) Show that the eigenvectors corresponding to distinct eigenvalues must the orthogonal to each other.
    c) Let $\mathbf{v}$ be an eigenvector of $\mathbb{A}$ and let $\mathbf{w}$ be perpendicular to $\mathbf{v}$. Show that $\mathbb{A}\mathbf{w}$ is also perpendicular to $\mathbf{v}$.

20. Evaluate the determinant $\begin{vmatrix} 1 & 2 & 2 & 1 \\ -1 & 2 & -2 & 1 \\ 1 & -2 & 2 & -1 \\ 2 & 1 & 2 & 1 \end{vmatrix}$.