MTH 306H MIDTERM II REVIEW PROBLEMS

0) Review your homework and study examples in the text.

1. Determine whether the given series converges. *Justify your answers*!

   a) $\sum_{n=1}^{\infty} \frac{2}{3^n}$.

   b) $\frac{1}{3} - \frac{2}{4} + \frac{3}{5} - \cdots + (-1)^{n-1}\frac{n}{n+2} + \cdots$.

   c) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$. (Hint: use the integral test. Recall: $\int \frac{dx}{x \ln x} = \ln \ln x + C$.)

   d) $\sum_{n=2}^{\infty} \left(\frac{1-3n}{3+4n}\right)^n$.

2. Consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$.

   a) The above series converges. Why?

   b) How many terms of the series do we need to add in order to find the sum of the series correct to three decimal places?

3. a) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} (-1)^n \frac{(x-3)^n}{n^3}$.

   b) What is the interval of convergence of the power series given in part a ?

4. Let $f(x) = \cos x$.

   a) Find the third degree Taylor polynomial $P_3(x)$ of $f(x)$ centered at $a = 0$.

   b) Write down an expression for the third degree remainder term $R_3(x)$ of $f(x)$ for $a = 0$, and show that if $|x| < 0.1$ then $|R_3(x)|$ is less than $5 \cdot 10^{-6}$.

5. a) What is the maximum error possible in using the approximation $\sin x \approx x - x^3/3! + x^5/5!$ if $0 \leq x \leq 0.3$?

   b) For what values of x is this approximation accurate to within 0.00005?

6. a) Find the Maclaurin series of $f(x) = \sqrt{1+x^3}$.
   b) Use your answer in part a) to find $f^{(12)}(0)$.

7. Find the *interval of convergence* of the series $\sum_{n=1}^{\infty} \dfrac{(2x-1)^n}{n^3}$.

8. Find a power series representation with $a = 0$ for the function $f(x) = x^2/(1-2x)^2$.

9. Let $P_n(x) = \sum_{k=0}^{n} \dfrac{x^k}{k!}$ be the n^{th} degree Taylor polynomial of the function $f(x) = e^x$ at $a = 0$. Show that $\lim_{n\to\infty} P_n(x) = e^x$ for all x.

10. Find the Taylor series expansion of the complex valued function $f(z) = (z-1)/(z-1-\mathrm{i})$ based at $a = \mathrm{i}$. Use your answer to compute the derivative $f^{(9)}(\mathrm{i})$.

11. Find the nth degree Taylor polynomial of $f(x) = \sin x$ centered at $a = \pi/4$. Find an expression for the remainder $R_n(x)$ and show that $\lim_{n\to\infty} R_n(x) = 0$ for all x. What do you conclude?

12. a) Find the formal Taylor series of the function $f(x) = \ln x$ centered at $x = 1$. What is the interval of convergence of the series? Does the series converge to the values of the function $f(x)$?
    b) Now replace x by the complex variable $z = x + \mathrm{i}y$ in the Taylor series in part a. Discuss the convergence of the resulting complex power series.
    c) The power series in part b in its domain of convergence defines a complex analytic function $\ln z$ which agrees with the real natural log-function when z is real. Recall that we defined a complex analytic exponential function e^z using power series. Is the function $\ln z$ the inverse of e^z also for complex z? Explain!

13. a) Find the power series expansion of the function $f(x) = 2x/(10+4x^4)$ at $a = 0$. Discuss the convergence of the resulting series. Compute the derivative $f^{(20)}(0)$.
    b) Find a power series expansion for $g(x) = \cos\sqrt{x}$ at $a = 0$.

14. a) Use the integral test to show that the series $\sum_{n=2}^{\infty} 1/(n(\ln n)^3)$ converges.
    c) How many terms would you have to add up the series to approximate the sum of the series within 10^{-4}?
    d) The approximation obtained in part b can be considerably improved. Explain!

15. Evaluate the improper integrals.

   a) $\int_{-\infty}^{\infty} \dfrac{x}{x^2+2}\, dx$.

   b) $\int_{-1}^{2} \dfrac{1}{(2x-1)^{1/3}}\, dx$.

16. Determine whether the given series converges. *Justify your answer*! If the series converges, also find its sum.

   a) $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \cdots + \dfrac{n}{n+1} + \ldots$.

   b) $\sum_{n=2}^{\infty} \dfrac{1}{\sqrt{n}\ln n}$.

   c) $\sum_{n=1}^{\infty} \dfrac{2}{n^2+2n}$.

   d) $\sum_{n=1}^{\infty} \dfrac{3^{n+1}}{\pi^n}$.

17. a) Use the integral test to show that the series $\sum_{n=2}^{\infty} 1/(n \ln n (\ln\ln n)^3)$ converges.

   b) How many terms would you have to add up the series to approximate the sum of the series within 10^{-4}?

   c) Next use the formula

$$T_N = S_N + \int_{N+1}^{\infty} f(x)$$

   given in the study guide to approximate the sum of the series. Find N so that the error is at most 10^{-4}.

18. a) Use the definition to find the formal Taylor series of the function $f(x) = \ln x$ centered at $x = 1$.

   b) What is the radius of convergence and the interval of convergence of the series?

   c) Does the series converge to the values of the function $f(x)$? Prove your claim.

   d) Now replace x by the complex variable $z = x + \mathrm{i}y$ in the Taylor series in part a). Discuss the convergence of the resulting complex power series.

19. a) Use the ratio test to show that the series $\sum_{n=0}^{\infty} (n^2+n)/2^n$ converges.

   b) Write $a_n = (n^2+n)/2^n$. Find N_o so that $a_{n+1}/a_n < 3/4$ for $n > N_o$.

c) Next use part b) to find an estimate for the error $|S - S_N|$, where $N \geq N_o$.

d) Finally find N so that S_N approximates the sum of the series within 0.001.

20. a) Find the limit

$$\lim_{x \to 0} \frac{\cos(x^2) - 1 + x^4/2 - x^8/24}{\exp(x^5) - 1 - x^5}.$$

b) Find exactly the sum of the series $\sum_{n \geq 1} \frac{n^2}{2^n}$.

c) Find a power series solution to the differential equation

$$y''(x) + x^2 y(x) = 0, \quad y(0) = 1, \quad y'(0) = 3.$$