35. Distributions with Compact Support

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Let Ω be an open subset of $\mathbb{R}^n$ and let $\mathrm{C}^\infty(\Omega)$ denote the space of infinitely differentiable functions on Ω. A linear functional u on $\mathrm{C}^\infty(\Omega)$ is continuous if and only if there exists a compact subset L of Ω, a constant $C > 0$ and an integer $m \geq 0$ such that

$$|<u, \phi>| \;\leq\; C \max_{|\alpha| \leq m} \sup_{x \in L} |D^\alpha \phi(x)|$$

for each $\phi \in \mathrm{C}^\infty(\Omega)$. The restriction of u to $\mathrm{C}_\mathrm{c}^\infty(\Omega)$ is then a distribution with $\operatorname{supp}(u) \subseteq L$. It is clear that the space of distributions with compact support may be identified with $\mathcal{E}'(\Omega)$, the dual space of $\mathrm{C}^\infty(\Omega)$.

Note each distribution with compact support has finite order. The smallest value of the integer m above is the order of u.

Theorem 1. *Let $u \in \mathcal{E}'(\Omega)$ have order m and support K. Then for each relatively open subset $\omega \subseteq \Omega$ with $K \subseteq \omega$ we have a constant C_ω such that*

$$|<u, \phi>| \;\leq\; C_\omega \max_{|\alpha| \leq m} \sup_{x \in \omega} |D^\alpha \phi(x)|$$

for each $\phi \in \mathrm{C}^\infty(\Omega)$.

Proof. By hypothesis there exists a compact subset L of Ω and a constant $C > 0$ such that

$$|<u, \phi>| \;\leq\; C \max_{|\alpha| \leq m} \sup_{x \in L} |D^\alpha \phi(x)|$$

for each $\phi \in \mathrm{C}^\infty(\Omega)$. Let $\chi \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$ be chosen such that $\chi = 1$ in a neighborhood of K. Then

$$\begin{aligned} |<u, \phi>| &= |<u, \chi\phi>| \\ &\leq C \max_{|\alpha| \leq m} \sup_{L} |D^\alpha(\chi\phi)| \\ &\leq C \max_{|\alpha| \leq m} \sup_{L \cap \omega} \left| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta}\chi D^\beta \phi \right| \\ &\leq C_\omega \max_{|\beta| \leq m} \sup_{\omega} \left| D^\beta \phi \right|. \end{aligned}$$

□

Note the supremum is taken over ω and the constant C_ω involves the derivatives of χ. Since χ is 1 on K and has support in ω we expect some of the derivatives of χ to be large on $\omega \sim K$ if ω is close to K. Thus we can not expect to obtain an estimate just on K. Nonetheless we can compute $<u, \phi>$ in terms of the m–jet of ϕ on K:

Theorem 2. *Let $u \in \mathcal{E}'(\Omega)$ have order m and support K. If $\phi \in \mathrm{C}^\infty(\Omega)$ and $D^\alpha\phi(x) = 0$ for each $|\,\alpha\,| \le m$ and $x \in K$, then $<u, \phi>= 0$.*

See Schwartz [3], theorem XXVIII(28), chapter III(3), section 7, Friedman [1], theorem 24, chapter 3, section 6, Donoghue [6], theorem in part II(2), section 21. In the case the support K is a regular set in the sense of Whitney [5] we can actually obtain $|<u_k, \phi>| \to 0$ if ϕ_k and its derivatives up to a certain order (depending on the *shape* of K) converge to 0 uniformly on K – see Schwartz [3], theorem XXXIV(34), chapter III(3), section 7 – but for a general K this result does not hold. See the example below. For related ideas see also Malgrange [2], especially chapter VII(7), and for a monograph concerning differentiable functions see Tougeron [4].

Proof. Since $u \in \mathcal{E}'(\Omega)$ has order m there exists a compact set L such that

$$|<u, \psi>| \;\le\; C \max_{|\,\alpha\,|\le m} \sup_L |\,D^\alpha\psi\,|$$

for each $\psi \in \mathrm{C}^\infty(\Omega)$. Note it follows that $K \subseteq L$. For each $\delta > 0$ let K_δ be the closed δ-neighborhood of K and let χ_δ be the characteristic function of K_δ. Let $\epsilon > 0$ be such that $3\epsilon < \mathrm{dist}(K, \partial\Omega)$. Let ρ be a Friedrichs' mollifier and let

$$\chi'_\epsilon = \rho_\epsilon * \chi_{2\epsilon}$$

so $\chi'_\epsilon \in \mathrm{C}^\infty_\mathrm{c}(\Omega)$, $\mathrm{supp}(\chi'_\epsilon) \subseteq K_{3\epsilon}$ and $\chi'_\epsilon = 1$ on K_ϵ.

Since $\chi'_\epsilon = 1$ in a neighborhood of K (namely K_ϵ) we have

$$<u, \psi>=<u, \chi'_\epsilon\psi>$$

and therefore

$$|<u, \psi>| \;\le\; C \max_{|\,\alpha\,|\le m} \sup_{L\cap K_{3\epsilon}} |\,D^\alpha\left(\chi'_\epsilon\psi\right)|$$

for any $\psi \in \mathrm{C}^\infty(\Omega)$.

If $x \in K_{3\epsilon}$ we can choose $y \in K$ with

$$|\,x - y\,| \;\le\; 3\epsilon.$$

Let

$$g(t) = D^\alpha\phi(y + t(x - y)), \quad 0 < t < 1,$$

so by Taylor's theorem with remainder

$$D^\alpha\phi(x) = g(1) = \sum_{k=0}^{m-|\,\alpha\,|} \frac{1}{k!} g^{(k)}(0) + \frac{1}{(m+1-|\,\alpha\,|)!} g^{(m+1-|\,\alpha\,|)}(t')$$

for some t' with $0 < t' < 1$, where $g^{(k)}(0)$ is a linear combination of $D^\beta \phi(y)$ for $\beta \geq \alpha$ and $|\,\beta\,| \leq m$. By hypothesis then $g^{(k)}(0) = 0$ since $y \in K$ and $|\,\beta\,| \leq m$. Thus

$$\begin{aligned} |\,D^\alpha \phi(x)\,| &\leq \frac{1}{(m+1-|\,\alpha\,|)!} \sup_{0<t<1} \left| g^{(m+1-|\,\alpha\,|)}(t) \right| \\ &\leq \frac{1}{(m+1-|\,\alpha\,|)!} \sup_{0<t<1} \left| D_t^{m+1-|\,\alpha\,|} \left(D^\alpha \phi(y + t(x-y)) \right) \right| \end{aligned}$$

Since $x \in K_{3\epsilon}$ and $y \in K$ we have

$$|\,D^\alpha \phi(x)\,| \leq C \max_{|\,\beta\,| \leq m+1} \sup_{K_{3\epsilon}} \left| D^\beta \phi \right| \epsilon^{m+1-|\,\alpha\,|}$$

which yields

$$|\,D^\alpha \phi(x)\,| \leq C'\, \epsilon^{m+1-|\,\alpha\,|}.$$

We have obtained this inequality for all $x \in K_{3\epsilon}$, $|\,\alpha\,| \leq m$, and all $\epsilon > 0$ with $3\epsilon < \mathrm{dist}(K, \partial\Omega)$. Now

$$D^\alpha \chi'_\epsilon(x) = \epsilon^{-|\,\alpha\,|} \int \chi_{2\epsilon}(x - \epsilon y)(D^\alpha \rho)(y)\ \mathrm{d}y$$

implies

$$|\,D^\alpha \chi'_\epsilon(x)\,| \leq C\, \epsilon^{-|\,\alpha\,|}$$

for $\epsilon > 0$. Thus by Leibniz' formula

$$\begin{aligned} |\,D^\alpha(\chi'_\epsilon \phi)\,| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\,D^\beta \chi'_\epsilon\,|\,|\,D^{\alpha-\beta}\phi\,| \\ &\leq C \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \epsilon^{m+1-|\,\beta\,|-|\,\alpha-\beta\,|} \end{aligned}$$

on $K_{3\epsilon}$. Hence we have $|\,<u, \phi>\,| \leq C\epsilon$ if $\epsilon > 0$ is small, that is, $<u, \phi> = 0$. ☐

Now here is the example promised above. It is taken from Schwartz [3], example at the end of chapter III(3), section 7.

Example 3. Let u_k be the distribution with compact support on $\mathbb{R}$ defined by

$$<u_k, \phi> = -k\, \phi(0) - \log(k)\, \phi'(0) + \sum_{h=0}^{k} \phi\left(\frac{1}{h}\right)$$

for $\phi \in \mathrm{C}^\infty(\mathbb{R})$. By Taylor's theorem the is $\psi \in \mathrm{C}^\infty(\mathbb{R})$ such that

$$\phi(x) = \phi(0) + \phi'(0)x + \psi(x)x^2.$$

It follows

$$<u_k, \phi> = \phi'(0) \left(\sum_{h=1}^{k} \frac{1}{h} - \log(k) \right) + \sum_{h=1}^{k} \psi\left(\frac{1}{h}\right) \frac{1}{h^2}.$$

The coefficient of $\phi'(0)$ converges to the Euler–Mascheroni constant and since $|\,\psi\,|$ is bounded on $[0,1]$ the last sum (Euler's series) converges. Hence by the Banach–Steinhaus theorem

$$<u,\phi>= \lim_{k\to\infty} <u_k,\phi>$$

is continuous, that is, defines a distribution u. Moreover it is clear that

$$\operatorname{supp}(u) \;=\; \left\{ x \;\mid\; x = 0, 1, \frac{1}{2}, \frac{1}{3}, \cdots \right\}.$$

If $\phi \in \mathrm{C}^\infty(\mathbb{R})$ and $\phi = 0$ on $\operatorname{supp}(u)$ and $\phi'(0) = 0$ then $<u,\phi>= 0$. This example illustrates theorem 1.

Now suppose we choose $\phi_k \in \mathrm{C}^\infty(\mathbb{R})$ such that $0 \le \phi_k \le \frac{1}{k}$ and

$$\phi_k(x) \;=\; \begin{cases} \frac{1}{\sqrt{k}} \text{ for } x \ge \frac{1}{k} \\ 0 \text{ for } x \le \frac{1}{k+1} \end{cases}$$

If $h \ge 1$ then the smoothness of ϕ_k implies $D^h\phi_k\left(\frac{1}{k+1}\right) = 0$ and $D^h\phi_k\left(\frac{1}{k}\right) = 0$. Thus by definition $D^h\phi_k = 0$ on $\operatorname{supp}(u)$ if $h \ge 1$. On the other hand $0 \le \phi_k(x) \le \frac{1}{\sqrt{k}}$ for all x. Thus $D^h\phi_k \to 0$ uniformly on $\operatorname{supp}(u)$ for each $h \ge 0$ as $k \to \infty$. On the other hand

$$<u,\phi_k>= \lim_{h\to\infty} \sum_{j=1}^{h} \phi_k\left(\frac{1}{j}\right) \;=\; \lim_{h\to\infty} \sum_{j=1}^{k} \frac{1}{\sqrt{k}} \;=\; \frac{k}{\sqrt{k}} \;=\; \sqrt{k}.$$

Thus the ϕ_k together with *all* their derivatives converge uniformly to 0 on $\operatorname{supp}(u)$ but $<u,\phi_k\; > \to \infty$ as $k \to \infty$.

Proposition 4. *Let* $a \in \mathbb{R}^n$, $u \in \mathcal{E}'(\mathbb{R}^n)$ *and suppose* $\operatorname{supp}(u) \subseteq \{a\}$. *Then there exist an integer* $m \ge 0$ *and constants* $c_\alpha \in \mathbb{C}$ *such that*

$$u = \sum_{|\,\alpha\,| \le m} c_\alpha D^\alpha \delta_a.$$

Proof. Since u has compact support it has finite order, say m. If $\phi_k \in \mathrm{C}^\infty(\mathbb{R}^n)$ then by Taylor's theorem

$$\phi(x) = \sum_{|\,\alpha\,| \le m} \frac{1}{\alpha!} D^\alpha\phi(a)(x-a)^\alpha + \psi(x)$$

where $\psi \in \mathrm{C}^\infty(\mathbb{R}^n)$ and $D^\beta\psi(a) = 0$ for $|\,\beta\,| \le m$. By theorem 1 it follows that $<u,\psi>= 0$. Thus $<u,\phi>= \sum_{|\,\alpha\,| \le m} c_\alpha D^\alpha \delta_a$ where

$$c_\alpha = \frac{(-1)^{|\,\alpha\,|}}{\alpha!} <u,(x-a)^\alpha> . \quad \Box$$

As you may expect there are similar structure theorems for *multiple layers*, that is, distributions supported by submanifolds. That is true. See Schwartz, [3].

Let me leave you with a little exercise to play with:

Exercise 1. *Find all solutions* $u \in \mathcal{D}'(\mathbb{R})$ *of the equation* $x^m u = 0$ *where* $m \geq 0$ *is an integer.*

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