

## # 15. Convolution of Functions

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If  $f$  and  $g$  are locally integrable functions on  $\mathbb{R}^n$  we define the *convolution product*  $f * g$  of  $f$  and  $g$  by

$$f * g(x) = \int f(x - y)g(y) dy$$

for those  $x$ , if any, for which the integral converges.

**Theorem 1 (MINKOWSKI'S Inequality).** *Let  $1 \leq p < \infty$  and let  $(X, \mathfrak{A}, dx)$  and  $(Y, \mathfrak{B}, dy)$  be  $\sigma$ -finite measure spaces. Let  $\phi$  be a complex valued  $\mathfrak{A} \times \mathfrak{B}$  measurable function on the product  $X \times Y$ . Then*

$$\left( \int \left| \int \phi(x, y) dy \right|^p dx \right)^{1/p} \leq \int \left( \int |\phi(x, y)|^p dx \right)^{1/p} dy$$

*in the sense that the integral on the left exists if the one on the right is finite, and in this case the inequality holds. Note the inequality may also be written as*

$$\left\| \int \phi(\cdot, y) dy \right\|_p \leq \int \|\phi(\cdot, y)\|_p dy.$$

*Proof.* If  $p = 1$  the theorem follows immediately from the FUBINI-TONELLI theorem. So assume  $p > 1$  and let

$$C = \int \left( \int |\phi(x, y)|^p dx \right)^{1/p} dy < \infty.$$

It follows that  $\int |\phi(x, y)|^p dx < \infty$  for almost all  $y$ . If  $(1/p) + (1/q) = 1$  and  $g \in \mathcal{L}^q(Y)$  then

$$\int |g(x)\phi(x, y)| dx \leq \|g\|_q \left( \int |\phi(x, y)|^p dx \right)^{1/p}$$

by HÖLDER'S inequality. Thus

$$\int \int |g(x)\phi(x, y)| dx dy \leq C \|g\|_q.$$

By the FUBINI–TONELLI theorem it follows that

$$\int |g(x)\phi(x, y)| dy < \infty$$

for almost all  $x$ . Since  $g \in \mathcal{L}^q(Y)$  is arbitrary we see that

$$\int |\phi(x, y)| dy < \infty$$

for almost all  $x$  and so  $h(x) = \int \phi(x, y) dy$  exists for almost all  $x$ . Now by the FUBINI–TONELLI theorem

$$\left| \int g(x)h(x) dx \right| \leq \int \int |g(x)\phi(x, y)| dy dx \leq C \|g\|_q.$$

It follows that there exists  $h' \in \mathcal{L}^p(Y)$  with  $\|h'\|_p \leq C$  such that

$$\int g(x)h(x) dx = \int g(x)h'(x) dx$$

for each  $g \in \mathcal{L}^q(Y)$ . It follows that  $h(x) = h'(x)$  for almost all  $x$  and therefore  $\|h\|_p \leq C$  which is what we wanted to prove.  $\square$

**Theorem 2 (YOUNG'S Inequality).** *If  $1 \leq p, q, r \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ ,  $f \in \mathcal{L}^p(\mathbb{R}^n)$  and  $g \in \mathcal{L}^q(\mathbb{R}^n)$  then*

$$f * g(x) = \int f(x - y)g(y) dy$$

*exists for almost all  $x$  and defines a function  $f * g \in \mathcal{L}^r(\mathbb{R}^n)$ . Moreover*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

*Proof.* If  $r = \infty$  then  $(1/p) + (1/q) = 1$ . Hence by HÖLDER'S inequality and the translation invariance of LEBESGUE measure

$$\int |f(x - y)g(y)| dy \leq \|f\|_p \|g\|_q.$$

Thus  $f * g(x)$  exists for each  $x$  and  $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$ .

Next suppose  $1 \leq r < \infty$ . Note that  $p \leq r$  and  $q \leq r$ . Let  $s = p(1 - 1/q) = 1 - p/r$  and note  $0 \leq s < 1$ . Let  $t = r/q$  and note that  $1 \leq t < \infty$ . Define  $q'$  by  $(1/q) + (1/q')$  and note  $1 < q' \leq \infty$ . Now let

$$h(x) = \int |f(x - y)g(y)| dy = \int |f(x - y)|^{1-s} |g(y)| |f(x - y)|^s dy.$$

By HÖLDER'S inequality we have

$$h(x) \leq \left( \int |f(x - y)|^{(1-s)q} |g(y)|^q dy \right)^{1/q} \| |f|^s \|_{q'}.$$

If  $s = 0$  then  $q = 1$ . If  $s \neq 0$  then  $sq' = p$ . In either case taking the  $q^{th}$  powers we obtain

$$h(x)^q \leq \int |f(x - y)|^{(1-s)q} |g(y)|^q dy \|f\|_p^{sq}.$$

Thus by MINKOWSKI'S inequality we have

$$\begin{aligned}
 \|h\|_{qt}^t &= \|h^q\|_t \\
 &\leq \|f\|_p^{sq} \left( \int \left( \int |f(x-y)|^{(1-s)q} |g(y)|^q dy \right)^t dx \right)^{1/t} \\
 &\leq \|f\|_p^{sq} \int \left( \int |f(x-y)|^{(1-s)qt} |g(y)|^{qt} dx \right)^{1/t} dy \\
 &= \|f\|_p^{sq} \|g\|_q^q \|f\|_{(1-s)qt}^{(1-s)q}.
 \end{aligned}$$

But  $qt = r$  and  $(1-s)r = p$ . Taking the  $q^{th}$  roots we obtain the inequality in the conclusion of the theorem.  $\square$

If  $f$  is a function on  $\mathbb{R}^n$  and  $a \in \mathbb{R}^n$  we define the *translation* of  $f$  by  $a$ ,  $\tau_a f$ , by

$$\tau_a f(x) = f(x - a).$$

If  $\phi \in \mathcal{L}^1(\mathbb{R}^n)$ ,  $\int \phi(x) dx = 1$  and  $\phi_\epsilon(x) = \epsilon^{-n} \phi(\epsilon^{-1}x)$  then the family of functions  $\phi_\epsilon$ ,  $\epsilon > 0$ , is called a *mollifier*. Note that  $\int \phi_\epsilon(x) dx = 1$ .

**Theorem 3.** Let  $\phi_\epsilon$  be a mollifier,  $1 \leq p < \infty$  and  $f \in \mathcal{L}^p(\mathbb{R}^n)$ . Then for each  $\epsilon > 0$

$$\|f * \phi_\epsilon - f\|_p \leq \int \|\tau_{\epsilon y} f - f\|_p |\phi(y)| dy.$$

*Proof.* Since  $\int \phi(x) dx = 1$  we have

$$f * \phi_\epsilon(x) - f(x) = \int (f(x - \epsilon y) - f(x)) \phi(y) dy.$$

Then by MINKOWSKI'S inequality

$$\begin{aligned}
 \|f * \phi_\epsilon - f\|_p &= \left( \int \left| \int (f(x - \epsilon y) - f(x)) \phi(y) dy \right|^p dx \right)^{1/p} \\
 &\leq \int \left( \int |f(x - \epsilon y) - f(x)|^p |\phi(y)|^p dx \right)^{1/p} dy \\
 &= \int \|\tau_{\epsilon y} f - f\|_p |\phi(y)| dy \quad \square
 \end{aligned}$$

**Corollary 4.** Let  $\phi \in \mathcal{L}^1(\mathbb{R}^n)$  and suppose  $\int \phi(x) dx = 0$ . If  $\epsilon > 0$  let  $\phi_\epsilon(x) = \epsilon^{-n} \phi(\epsilon^{-1}x)$ . Then for  $1 \leq p < \infty$  and  $f \in \mathcal{L}^p(\mathbb{R}^n)$  we have

$$\|f * \phi_\epsilon\|_p \leq \int \|\tau_{\epsilon y} f - f\|_p |\phi(y)| dy.$$

As an example of a mollifier define

$$\phi(x) = \begin{cases} c(1 - |x|) & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

where  $c$  is chosen so that  $\int \phi(x) dx = 1$ . Then  $\phi_\epsilon$  is a continuous mollifier and  $\phi_\epsilon(x) = 0$  if  $|x| \geq \epsilon$ .

**Proposition 5.** Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ . Let  $(U_j)_{j \in J}$  be a collection of open subsets of  $\Omega$  with union  $U$ . Let  $E \subseteq U$ . If  $E \cap U_j$  is a set of LEBESGUE measure 0 for each  $j$  then  $E$  has measure 0.

*Proof.* Let  $Q$  be the countable set consisting of all open balls in  $\mathbb{R}^n$  with rational radius and rational center coordinates. Then

$$U_j = \bigcup \{q \mid q \in Q, q \subset U_j\}$$

implies  $E$  is a countable union of sets of measure 0 of the form  $E \cap q$ . □

Note it is essential that the  $U_j$  above be open.

If  $f \in \mathcal{L}^1(\Omega, \text{loc})$  then by the proposition above there is a largest *open* subset  $U$  of  $\Omega$  on which  $f$  is 0 almost everywhere - just take the union of open sets on which  $f$  vanishes almost everywhere. The complement of  $U$  in  $\Omega$  is called the *support* of  $f$  in  $\Omega$  and is denoted by  $\text{supp}(f)$ .

**Proposition 6.** If  $f: \Omega \rightarrow \mathbb{C}$  is continuous then

$$\text{supp}(f) = \overline{\{x \in \Omega \mid f(x) \neq 0\}}.$$

*Proof.* Exercise. □

If  $\Omega$  is a topological space we denote by  $C_c(\Omega)$  the set of complex valued continuous functions on  $\Omega$  each vanishing outside of some compact set. Such functions are said to have *compact support*. If  $\Omega$  is a smooth manifold we denote by  $C_c^\infty(\Omega)$  the infinitely often continuously differentiable functions with compact support in  $\Omega$ .

**Lemma 7.** Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and let  $K$  be a compact subset of  $\Omega$ . Then there exists  $\psi \in C_c(\Omega)$  such that  $0 \leq \psi \leq 1$  and  $\psi = 1$  on  $K$ .

We give two proofs.

*Proof. 1.* Let  $\omega$  be a relatively compact neighborhood of  $K$  with  $\bar{\omega} \subseteq \Omega$ . By URYSOHN'S lemma there is  $\psi \in C(\mathbb{R}^n)$  such that  $0 \leq \psi \leq 1$ ,  $\psi = 1$  on  $K$  and  $\psi = 0$  outside  $\omega$ .

*Proof.* 2. Let  $\phi_\epsilon$  be a continuous mollifier chosen as above and let

$$\begin{aligned}\psi(x) &= \int \chi_L(x-y)\phi_\epsilon(y) dy \\ &= \int \chi_L(y)\phi_\epsilon(x-y) dy = \int_L \phi_\epsilon(x-y) dy\end{aligned}$$

where  $L$  is the closed  $\delta$ -neighborhood of  $K$  and  $3\delta = \text{dist}(K, \partial\Omega)$ . If  $0 < \epsilon < \delta$  then  $\phi = 1$  on  $K$  and the other properties are obvious.  $\square$

By introducing a smooth mollifier in place of the continuous one above, the second proof may be used to show the existence of numerous smooth functions with compact support.

**Theorem 8 (Density of  $C_c(\Omega)$  in  $\mathcal{L}^p(\Omega)$ ).** *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and let  $1 \leq p < \infty$ . Then  $C_c(\Omega)$  is dense in  $\mathcal{L}^p(\Omega)$ .*

*Proof.* We denote the LEBESGUE measure of a BOREL set  $B$  by  $|B|$ . Since the simple functions are dense in  $\mathcal{L}^p(\Omega)$  for  $p$  finite it suffices to show that we can approximate the characteristic function  $\chi_A$  of a Borel set  $A$  of finite measure by a continuous function with compact support. Let  $\epsilon > 0$ . By regularity of LEBESGUE measure there is a compact set  $K \subseteq A$  and an open set  $U \supseteq A$  such that  $|U \setminus K| < \epsilon^p$ . By the lemma there is  $\phi \in C_c(U)$  with  $0 \leq \phi \leq 1$  such that  $\psi = 1$  on  $K$ . Now  $|\chi_A - \phi| \leq \chi_U - \chi_K$  implies

$$\|\chi_A - \phi\|_p \leq |U \setminus K|^{1/p} < \epsilon. \quad \square$$

One can also base a proof on LUSIN's theorem that a measurable function with support with finite measure coincides outside a set of arbitrarily small measure with a continuous function. See [2], p. 69 for the argument and p. 55 for LUSIN's theorem.

**Theorem 9 (Continuity of Translation in  $\mathcal{L}^p$ ).** *Let  $1 \leq p < \infty$ ,  $f \in \mathcal{L}^p(\mathbb{R}^n)$  and define*

$$\theta: \mathbb{R}^n \rightarrow \mathcal{L}^p(\mathbb{R}^n)$$

*by  $\theta(y) = \tau_y f$ . Then  $\theta$  is uniformly continuous.*

*Proof.* Let  $\epsilon > 0$ . Choose  $g \in C_c(\Omega)$  such that  $\|f - g\|_p < \epsilon/3$ . If  $y, z \in \mathbb{R}^n$  and  $v = y - z$  then

$$\begin{aligned}\|\tau_y f - \tau_z f\|_p &\leq \|\tau_y f - \tau_y g\|_p + \|\tau_y g - \tau_z g\|_p + \|\tau_z g - \tau_z f\|_p \\ &\leq \frac{2}{3}\epsilon + \|\tau_y g - \tau_z g\|_p \\ &\leq \frac{2}{3}\epsilon + \|\tau_v g - g\|_p\end{aligned}$$

by translation invariance of LEBESGUE measure. Since  $g$  has compact support the support of  $\tau_v g$  stays in a fixed compact set  $K$  for  $|v| \leq 1$ . Since  $g$  is bounded we have

$$|\tau_v g| \leq M \chi_K \text{ for } |v| \leq 1.$$

It follows that

$$|\tau_v g - g|^p \leq (2M)^p \chi_K \in \mathcal{L}^1(\mathbb{R}^n) \quad \text{for} \quad |v| \leq 1.$$

Since  $g$  is continuous we have  $\tau_v g - g \rightarrow 0$  pointwise. By the dominated convergence theorem  $\int |\tau_v g - g|^p dx \rightarrow 0$  as  $v \rightarrow 0$  through any sequence (and so as  $v \rightarrow 0$ ). Thus there is  $\delta > 0$  such that  $0 < \delta \leq 1$  and

$$\|\tau_v g - g\|_p < \frac{1}{3}\epsilon \quad \text{if} \quad |v| < \delta. \quad \square$$

**Corollary 10 (Approximate identity).** Let  $\phi \in \mathcal{L}^1(\mathbb{R}^n)$ ,  $1 \leq p < \infty$  and let  $f \in \mathcal{L}^p(\mathbb{R}^n)$ . If  $\int \phi(x) dx = 1$  then  $f * \phi_\epsilon \rightarrow f$  in  $\mathcal{L}^p(\mathbb{R}^n)$  as  $\epsilon \rightarrow 0$ . If  $\int \phi(x) dx = 0$  then  $f * \phi_\epsilon \rightarrow 0$  in  $\mathcal{L}^p(\mathbb{R}^n)$  as  $\epsilon \rightarrow 0$ .

*Proof.* For the first case

$$\|f * \phi_\epsilon - f\|_p \leq \int \|\tau_{\epsilon y} f - f\|_p |\phi(y)| dy.$$

The integrand is bounded by  $2\|f\|_p |\phi| \in \mathcal{L}^1(\mathbb{R}^n)$  and goes to 0 as  $\epsilon \rightarrow 0$  by continuity of translation. Thus the dominated convergence theorem yields the desired result. The second part is similar.  $\square$

**Exercise 1.** If  $\phi$  above has compact support and is bounded we can avoid using the dominated convergence theorem. See the proof of theorem 12. Then if in addition  $f$  is continuously differentiable we can obtain a LIPSCHITZ' type estimate  $\|f * \phi_\epsilon - f\|_p \leq C\epsilon$  by using the mean value theorem. Try it.

**Corollary 11.** If  $1 < p < \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $f \in \mathcal{L}^p(\mathbb{R}^n)$  and  $g \in \mathcal{L}^q(\mathbb{R}^n)$  then  $f * g$  is uniformly continuous.

*Proof.*

$$f * g(x) - f * g(z) = \int (\tau_{-x} f - \tau_{-z} f)(-y)g(y) dy.$$

implies

$$|f * g(x) - f * g(z)| \leq \|\tau_{-x} f - \tau_{-z} f\|_p \|g\|_q. \quad \square$$

*Example 1.* A function  $h$  on  $\mathbb{R}^n$  is said to be of *positive type* if for each set of points  $x_1, \dots, x_N \in \mathbb{R}^n$  the matrix  $(h(x_i - x_j))$  is of positive type, that is, for each  $z \in \mathbb{C}^n$

$$\sum_{j,k} h(x_j - x_k) z_j \bar{z}_k \geq 0.$$

Now if  $f \in \mathcal{L}^2(\mathbb{R}^n)$  define  $h$  by

$$h(x) = (\tau_{-x} f | f) = \int f(x+y)\overline{f(y)} = f * \bar{f}^\vee$$

where  $g^\vee(x) = g(-x)$ . By the corollary  $h$  is bounded and uniformly continuous. If  $x_1, \dots, x_N \in \mathbb{R}^n$  and  $z \in \mathbb{C}^N$  define

$$g_z(x) = \sum_j f(x_j + x) z_j.$$

Then

$$\begin{aligned}\|g_z\|_2^2 = (g_z | g_z) &= \sum_{j,k} \int f(x_j + x) \overline{f(x_k + x)} dx z_j \overline{z_k} \\ &= \sum_{j,k} h(x_j - x_k) z_j \overline{z_k}.\end{aligned}$$

Thus  $h$  is of positive type. The function  $h$  is called the **PATTERSON transform** of  $f$  (in crystallography). See **BUERGER** [1]. It represents what can be reconstructed from x-ray diffraction data when  $f$  is the electron density in a crystal. The significance of the **PATTERSON transform** is best understood in terms of the **FOURIER transform**.

*Example 2.* Let

$$\phi(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1 + |x|^2)^{\frac{n+1}{2}}}.$$

Then  $P(x, \epsilon) = \phi_\epsilon(x)$  is a mollifier called the **ABEL-POISSON kernel**. Explicitly

$$P(x, \epsilon) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{\epsilon}{(\epsilon^2 + |x|^2)^{\frac{n+1}{2}}}$$

and for any  $f \in \mathcal{L}^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$

$$\int P(x - y, \epsilon) f(y) dy \rightarrow f(x)$$

in  $\mathcal{L}^p(\mathbb{R}^n)$  as  $\epsilon \rightarrow 0$ .

Note if  $n = 1$  then  $P(x, \epsilon) = \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2}$  is the well-known **POISSON kernel of the upper half plane** in  $\mathbb{R}^2$ .

*Example 3.* Let

$$\phi(x) = \pi^{-n/2} e^{-|x|^2}.$$

Then

$$W(x, \epsilon) = \phi_{2\sqrt{\epsilon}}(x) = (4\pi\epsilon)^{-n/2} e^{-|x|^2/4\epsilon}$$

is called the **GAUSS-WEIERSTRASS kernel**. If  $1 \leq p < \infty$  and  $f \in \mathcal{L}^p(\mathbb{R}^n)$  then

$$\int W(x - y, \epsilon) f(y) dy \rightarrow f(x)$$

in  $\mathcal{L}^p(\mathbb{R}^n)$  as  $\epsilon \rightarrow 0$ .

One important application of the convolution product is *regularization* of functions, that is, the approximation of functions by smooth functions. Let

$$u(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Since  $t^{-k}e^{-1/t} \rightarrow 0$  as  $t \rightarrow 0$  we see that  $u \in C^\infty(\mathbb{R})$ . Now let

$$\rho(x) = cu(1 - |x|^2), \quad x \in \mathbb{R}^n.$$

Then  $\rho \in C^\infty(\mathbb{R}^n)$  and  $\rho(x) = 0$  if  $|x| \geq 1$ . Moreover, for a suitable choice of the constant  $c$  we have  $\rho(x) \geq 0$  and  $\int \rho(x) dx = 1$ . As usual we let

$$\rho_\epsilon(x) = \epsilon^{-n} \rho(\epsilon^{-1}x).$$

Then

1.  $\rho_\epsilon \in C_c^\infty(\mathbb{R}^n)$
2.  $\text{supp } \rho_\epsilon = \{x \in \mathbb{R}^n \mid |x| \leq \epsilon\}$
3.  $\rho_\epsilon(x) \geq 0$
4.  $\int \rho_\epsilon(x) dx = 1$ .

We will refer to  $\rho_\epsilon$ , satisfying these four properties, as a FRIEDRICHS' mollifier.

**Theorem 12.** Let  $\rho_\epsilon$  be a Friedrichs' mollifier. If  $f \in \mathcal{L}^1(\mathbb{R}^n, \text{loc})$  then the convolution

$$f * \rho_\epsilon(x) = \int f(x-y)\rho_\epsilon(y) dy = \int f(x-\epsilon y)\rho(y) dy$$

exists for each  $x \in \mathbb{R}^n$ . Moreover

1.  $f * \rho_\epsilon \in C^\infty(\mathbb{R}^n)$
2.  $\text{supp}(f * \rho_\epsilon) \subseteq \text{closed } \epsilon\text{-neighborhood of } \text{supp } f$ .
3. if  $1 \leq p < \infty$  and  $f \in \mathcal{L}^p(\mathbb{R}^n)$  then  $f * \rho_\epsilon \rightarrow f$  in  $\mathcal{L}^p(\mathbb{R}^n)$  as  $\epsilon \rightarrow 0$ . Indeed  $\|f - f * \rho_\epsilon\|_p \leq \sup_{|y| \leq \epsilon} \|f - \tau_y f\|_p$ .
4. if  $K$  is a compact set of points of continuity of  $f$  then  $f * \rho_\epsilon \rightarrow f$  uniformly on  $K$  as  $\epsilon \rightarrow 0$ .

*Proof.* The convolution exists for each  $x$  because the mollifier has compact support. Note  $f * \rho_\epsilon(x) = \int \rho_\epsilon(x-y)f(y) dy$  yields  $f * \rho_\epsilon \in C^\infty(\mathbb{R}^n)$  by standard results on differentiating under the integral sign (since  $\rho$  has compact support). The second statement is obvious and the third follows from

$$\|f - f * \rho_\epsilon\|_p \leq \int \|\tau_{\epsilon y} f - f\|_p \rho(y) dy \leq \sup_{|y| \leq \epsilon} \|\tau_y f - f\|_p$$

where we used that  $\rho$  has support in the unit ball. Let  $K$  be a compact set of points of continuity of  $f$ . The standard calculus argument that  $f$  is uniformly continuous on  $K$  actually shows a little bit more: let  $\eta > 0$ , then there is  $\delta > 0$  such that if  $x \in K$ ,  $z \in \mathbb{R}^n$  and  $|x - z| < \delta$  then it follows that  $|f(x) - f(z)| < \eta$ . Note we do not require  $z$  to be in  $K$ . Now

$$f(x) - f * \rho_\epsilon(x) = \int_{|y| \leq \epsilon} (f(x) - f(x-y)) \rho_\epsilon(y) dy.$$

Hence if  $0 < \epsilon < \delta$  then

$$\begin{aligned} |f(x) - f * \rho_\epsilon(x)| &\leq \int_{|y| \leq \epsilon} |f(x) - f(x-y)| \rho_\epsilon(y) dy \\ &\leq \eta \int \rho_\epsilon(y) dy \\ &= \eta \end{aligned}$$

for each  $x \in K$ . □

**Corollary 13.** *If  $\Omega$  is an open subset of  $\mathbb{R}^n$  and  $K$  is a compact subset of  $\Omega$  then there exists  $\phi \in C_c^\infty(\Omega)$  with  $0 \leq \phi \leq 1$  such that  $\phi = 1$  on  $K$ .*

Note theorem 16 below, the finite partition of unity theorem, is a useful generalization of corollary 13.

*Proof.* Let  $\delta = \frac{1}{3} \text{dist}(K, \partial\Omega) > 0$  and let  $f$  be the characteristic function of the closed  $\delta$ -neighborhood of  $K$  and let  $0 < \epsilon < \delta$ . Then  $\phi = f * \rho_\epsilon \in C^\infty(\mathbb{R}^n)$  has support in the closed  $2\delta$ -neighborhood of  $K$  and so has compact support in  $\Omega$ . Clearly  $0 \leq \phi \leq 1$  and  $\phi = 1$  on the  $(\delta - \epsilon)$ -neighborhood of  $K$ .  $\square$

Corollary 13 and a slight modification of the (first) proof of theorem 8 yield easily

**Theorem 14 (Density of  $C_c^\infty(\Omega)$  in  $\mathcal{L}^p(\Omega)$ ).** *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and let  $1 \leq p < \infty$ . Then  $C_c^\infty(\Omega)$  is dense in  $\mathcal{L}^p(\Omega)$ .*

The assertion in theorem 12 that the support of  $f * \rho_\epsilon$  is contained in the closed  $\epsilon$ -neighborhood of  $\text{supp } f$  is special for the Friedrichs' mollifier. In general

$$\text{supp } f * \phi_\epsilon \subseteq \text{supp } f + \text{supp } \phi_\epsilon$$

if  $\phi_\epsilon$  is a mollifier with compact support. For example, let  $\Gamma$  be a closed convex cone in  $\mathbb{R}^n$  with nonempty interior  $\Gamma^\circ$ . Let  $\Omega = \{x \in \Gamma^\circ \mid |x| < 1\}$ . By the corollary there is a nontrivial  $\phi \in C_c^\infty(\Omega)$  with  $0 \leq \phi$ . Multiplying  $\phi$  by a positive constant if necessary we may arrange to have  $\int \phi(x) dx = 1$ . Now suppose  $f \in \mathcal{L}^1(\mathbb{R}^n, \text{loc})$  and  $\text{supp } f \subseteq \Gamma$ . Then  $\text{supp } f * \phi_\epsilon \subseteq \Gamma$ , that is, we can approximate  $f$  by smooth functions with support in  $\Gamma$ . An important special case is the case where  $\Gamma$  is a closed half-space.

Corollary 13 may be used to show the existence of *smooth partitions of unity*.

**Theorem 15 (Partition of unity).** *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and let  $U_j, j \in J$ , be a locally finite open cover of  $\Omega$  such that each  $U_j$  has compact closure in  $\Omega$ . Then there exist  $\phi_j$  such that*

$$\phi_j \in C_c^\infty(U_j), \quad \phi_j \geq 0 \quad \text{and} \quad \sum_{j \in J} \phi_j(x) = 1 \text{ for each } x \in \Omega.$$

*Proof.* By the shrinking lemma for  $T_4$  spaces there exist open sets  $\omega_j$  which form a covering of  $\Omega$  and which satisfy  $\overline{\omega_j} \subseteq U_j$  for each  $j \in J$ . Now choose  $\psi_j \in C_c^\infty(U_j)$  such that  $0 \leq \psi_j \leq 1$  and  $\psi_j = 1$  on  $\overline{\omega_j}$ . The sum  $\psi(x) = \sum_j \psi_j(x)$  is locally finite and bounded below by 1. Thus  $\psi \in C^\infty(\Omega)$  and  $1 \leq \psi$ . Now let  $\phi_j = \psi_j / \psi$ .  $\square$

We call  $\phi_j, j \in J$ , a *smooth partition of unity* subordinate to the locally finite open cover  $U_j, j \in J$ .

Partitions of unity are used to glue local objects together to obtain a global object. If we can make do with a *finite partition of unity* in a neighborhood of a compact set, we can give an argument which does not involve division (and so may be useful in a more general context).

**Theorem 16 (Finite partition of unity).** Let  $K$  be a compact subset of  $\mathbb{R}^n$  and let  $(U_j)_{j=1,2,\dots,N}$  be a finite open cover of  $K$ . Then there exist functions  $\phi_j \in C_c^\infty(U_j)$  such that  $\phi_j \geq 0$  for each  $j = 1, 2, \dots, N$  and

$$\sum_{j=1}^N \phi_j = 1$$

in a neighborhood of  $K$ .

*Proof.* (See [2], p. 40). For each  $x \in K$  let  $V_x$  be an open neighborhood of  $x$  such that  $\overline{V_x}$  is a compact subset of each  $U_j$  with  $x \in U_j$ . Since  $K$  is compact there is a finite set  $x_1, x_2, \dots, x_m$  in  $K$  such that

$$K \subseteq \bigcup_{k=1}^m V_{x_k}.$$

Now for each  $j$  let  $K_j$  be the union of those  $\overline{V_{x_k}}$  which are contained in  $U_j$ . Then  $K_j$  is compact,  $K_j \subseteq U_j$  and

$$K \subseteq K_1 \cup \dots \cup K_N.$$

By corollary 13 we may choose  $\psi_j \in C_c^\infty(U_j)$  so  $0 \leq \psi_j \leq 1$  in a neighborhood of  $K_j$ . Finally let

$$\begin{aligned} \phi_1 &= \psi_1 \\ \phi_2 &= (1 - \psi_1) \psi_2 \\ \phi_3 &= (1 - \psi_1)(1 - \psi_2) \psi_3 \\ &\dots \\ \phi_N &= (1 - \psi_1)(1 - \psi_2) \dots (1 - \psi_{N-1}) \psi_N \end{aligned}$$

Clearly  $0 \leq \phi_j$  for each  $j$  and

$$\phi_1 + \dots + \phi_N = 1 - (1 - \psi_1)(1 - \psi_2) \dots (1 - \psi_N).$$

Since

$$K \subseteq \bigcup_{j=1}^N K_j$$

for each  $x \in K$  there is  $j$  so that  $\psi_j(x) = 1$ . Thus  $\phi_1 + \dots + \phi_N = 1$  on  $K$ . To obtain the equality on a neighborhood of  $K$  we would start by enlarging  $K$  a bit first.  $\square$

As another application of regularization by convolution we will prove an important theorem of DU BOIS-REYMOND (which is related to the so called fundamental lemma of the calculus of variations). This theorem allows us to identify (equivalence classes of) locally integrable functions with distributions.

**Theorem 17 (DU BOIS-REYMOND).** Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ . If  $f \in \mathcal{L}^1(\Omega, loc)$  and  $\int f(x)\phi(x) dx = 0$  for each  $\phi \in C_c^\infty(\Omega)$  then  $f = 0$  almost everywhere in  $\Omega$ .

*Proof.* Let  $\omega$  be a relatively compact subset of  $\Omega$ . Choose  $\delta$  so that  $0 < 2\delta < \text{dist}(\omega, \partial\Omega)$ . Let  $K$  be the closed  $\delta$ -neighborhood of  $\omega$  so  $K$  is a compact subset of  $\Omega$ . Now let  $\psi = \xi_K$  so  $\psi f \in \mathcal{L}^1(\mathbb{R}^n)$  (after extending  $f$  to  $\mathbb{R}^n$  by 0). Let  $\rho_\epsilon$  be a Friedrichs' mollifier. If  $x \in \omega$  and  $0 < \epsilon < \delta$  then

$$\begin{aligned} (\psi f) * \rho_\epsilon(x) &= \int (\psi f)(x - y) \rho_\epsilon(y) dy \\ &= \int f(x - y) \rho_\epsilon(y) dy \\ &= \int f(y) \rho_\epsilon(x - y) dy \\ &= 0 \end{aligned}$$

since  $x \in \omega$ ,  $0 < \epsilon < \delta$  imply  $x - y \in K$  which implies  $\psi(x - y) = 1$  and also  $y \rightarrow \rho_\epsilon(x - y) \in C_c^\infty(\Omega)$ . Now by the approximate identity theorem (theorem 10) we have  $(\psi f) * \rho_\epsilon \rightarrow \psi f$  in  $\mathcal{L}^1(\mathbb{R}^n)$ . Thus  $\psi f = 0$  almost everywhere, and so  $f = 0$  almost everywhere in  $\omega$ . Since  $\Omega$  is a countable union of sets satisfying the hypotheses assumed for  $\omega$  we have  $f = 0$  almost everywhere in  $\Omega$ .  $\square$

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