

80. Example. Iterated Function Systems

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In this section we describe an application of Banach's Contraction Mapping Principle, [1], to the study of iterated function systems and their corresponding fractals. For a large number of examples and many of the details not presented here see the book by M. Barnsley, [2].

Let X be a metric space. If B is a nonempty subset of X we define

$$d(x, B) = \inf_{y \in B} d(x, y).$$

Let $x, y \in X$. If $\epsilon > 0$ choose $z \in B$ such that $d(y, z) < d(y, B) + \epsilon$. Then $d(x, B) \leq d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + d(y, B) + \epsilon$. Thus we conclude

$$|d(x, B) - d(y, B)| \leq d(x, y), \quad x, y \in X.$$

It follows that $x \rightarrow d(x, B)$ is a continuous nonnegative function vanishing precisely on \overline{B} .

Now if A and B are nonempty subsets of X we define

$$d(A, B) = \sup_{x \in A} d(x, B).$$

Obviously $0 \leq d(A, B) \leq \infty$. Let A, B, C be nonempty subsets of X . Let $x \in A$ and let $\epsilon > 0$. Choose $z \in C$ such that $d(x, z) < d(x, C) + \epsilon$. Now $d(x, B) \leq d(x, z) + d(z, B) \leq d(x, C) + d(z, B) + \epsilon$. Since $z \in C$ we have $d(z, B) \leq d(C, B)$ and therefore $d(x, B) \leq d(x, C) + d(C, B) + \epsilon$. Taking the supremum over $x \in A$ we obtain

$$d(A, B) \leq d(A, C) + d(C, B).$$

We have to be careful with the order here:

Exercise 80.1. Find an example with $d(A, B) \neq d(B, A)$.

Let $\mathfrak{K}(X)$ be the set of nonempty compact subsets of X . If $A, B \in \mathfrak{K}(X)$ we define the *Hausdorff metric* d_H by

$$d_H(A, B) = \max \{d(A, B), d(B, A)\}.$$

If A is a subset of X we define the *closed ϵ -neighborhood* A_ϵ of A by

$$A_\epsilon = \{y \in X \mid d(x, y) \leq \epsilon \text{ for some } x \in A\}.$$

Exercise 80.2. If $A, B \in \mathfrak{K}(X)$ and $\epsilon > 0$ then $d(A, B) \leq \epsilon$ if and only if $A \subseteq B_\epsilon$.

Exercise 80.3. d_H is a metric on $\mathfrak{K}(X)$.

Theorem 80.1. *Completeness of $\mathfrak{K}(X)$.* If X is a complete metric space then $\mathfrak{K}(X)$ is complete in the Hausdorff metric. If $(A_n)_{n \geq 1}$ is a Cauchy sequence in $\mathfrak{K}(X)$ and $A = \lim_{n \rightarrow \infty} A_n$ then

$$A = \left\{ x \in X \mid \text{there exists } x_n \in A_n \text{ so that } \lim_{n \rightarrow \infty} x_n = x \right\}.$$

For a proof of this theorem see [2].

Now let T_j be contraction mappings of X with contractivity coefficients c_j , $j = 1, \dots, n$. Define $T: \mathfrak{K}(X) \rightarrow \mathfrak{K}(X)$ by

$$T(A) = T_1(A) \cup \dots \cup T_n(A), \quad A \in \mathfrak{K}(X).$$

One can show that T is a contraction mapping with contractivity coefficient $c = \max_j c_j$. Thus T has a unique fixed point A in $\mathfrak{K}(X)$ and we can approximate it by computing iterates of T applied to an arbitrary element of $\mathfrak{K}(X)$. The system $(T_j)_{1 \leq j \leq n}$ is called an *iterated function system*, abbreviated as IFS. The fixed point A is called the *fractal* determined by the IFS.

Exercise 80.4. Let $X = [0, 1]$ and consider the IFS $\{T_1, T_2\}$ defined by $T_1(t) = t/3$ and $T_2(t) = 2/3 + t/3$. Show by a direct calculation that the Cantor set is the fixed point of this IFS.

References

- [1] Stefan Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3:133–181, 1922.
- [2] Michael Barnsley. *Fractals everywhere*. Academic Press, Boston · New York · London, 1988.

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