We have obtained a number of results but here I will mention only the highlights.

We started by discussing power series and the radius of convergence of a power series. We saw that the sum of a convergent power series is analytic in the disk of convergence. Furthermore, we saw that a convergent power series may be differentiated term–by–term.

We have obtained primitive versions of the Cauchy theorem and the Cauchy Integral theorem. As we shall see, these theorems will suffice to obtain many important properties of analytic functions.

Theorem 1 (Primitive Cauchy theorem) *If Ω is an open rectangle, any orientation, or an open disk, γ is a piecewise continuously differentiable closed contour in Ω, and $f : \Omega \to \mathbb{C}$ is analytic then*

$$\int_\gamma f(z)\, \mathrm{d}z = 0.$$

Note this theorem is equivalent to asserting that f has a primitive in Ω.

Theorem 2 (Primitive Cauchy Integral theorem) *If Ω is an open rectangle, any orientation, or an open disk, $a \in \Omega$, γ is a piecewise continuously differentiable closed contour in Ω, $f : \Omega \to \mathbb{C}$ is analytic, and $a \notin \mathrm{traj}(\gamma)$ then*

$$\nu(\gamma, a)\, f(a) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(z)}{z-a}\, \mathrm{d}z$$

where $\nu(\gamma, a)$ is the index or winding number of γ about a and is given by

$$\nu(\gamma, a) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\mathrm{d}z}{z-a}.$$

The Cauchy Integral theorem is made useful by the following fact:

Proposition 3 *If γ is the boundary of the open disk $D(z_0, r)$, say $\gamma(t) = z_0 + r\mathrm{e}^{\mathrm{i}t}$ then*

$$\nu(\gamma, z) = \begin{cases} 1 & \text{if } z \in D(z_0, r) \\ 0 & \text{if } |\, z - z_0 \,| > r. \end{cases}$$

Starting this week, week 4, we will show that a function f analytic in an open disk $D(z_0, r)$ is the sum of a convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n\, (z - z_0)^n$$

in the disk. Note it follows that the series has radius of convergence $\geq r$ and

$$c_n = \frac{1}{n!}\, f^{(n)}(z_0)\,.$$

Thus a function f is analytic in an open set Ω if and only if for each $z_0 \in \Omega$, f is the sum of a convergent power series in powers of $z - z_0$ in $D(z_0, r)$ where $r = \mathrm{dist}(z_0, \partial\Omega)$.

*Bent Petersen File ref: 611s2003˙sum1.tex