**Instruction:** There are 16 problems below. Choose 3 to enjoy and for which to turn in solutions. You do not have to limit yourself to the “easy” problems. If you turn in more than 3 be sure to indicate which three I should grade.

There may be some errors in the statements of the problems. Part of your job is to correct any such errors. As usual you may discuss the problems with anyone, but writing up your solutions should be a solitary activity.

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**Problem 1.** An entire function $f$ is said to be of *exponential type* if there exist real numbers $t$ and $A_t$ such that

$$|\, f(z) \,| \leq A_t \, \mathrm{e}^{t|\, z \,|}, \quad z \in \mathbb{C}.$$

The *type* $\tau$ of $f$ is the infemum of such $t$, so a polynomial, for example, has type $0$. If $M(r) = \max_{|\, z \,|=r} |\, f(z) \,|$ then clearly

$$\tau = \limsup_{r \to \infty} r^{-1} M(r).$$

Now consider a power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and suppose

$$\limsup_{n \to \infty} n \,|\, c_n \,|^{\frac{1}{n}} < \infty.$$

Show that $f$ is entire, and moreover, $f$ is of exponetial type

$$\tau = \mathrm{e}^{-1} \limsup_{n \to \infty} n \,|\, c_n \,|^{\frac{1}{n}}.$$

Conversely for any entire function of exponential type this formula gives the type.

**Hint:** If $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is an entire function the Cauchy inequalities imply

$$|\, c_n \,| r^n \leq M(r) \leq \sum_{n=0}^{\infty} |\, c_n \,| r^n.$$

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**Problem 2.** Let $R > 0$ and let $f$ be analytic in the disk $D(0, R)$. Suppose $|\, f(z) \,| \leq M$ in $D(0, R)$ and there is an integer $n \geq 1$ such that

$$f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0.$$

Then

$$|\, f(z) \,| \leq M \left( \frac{|\, z \,|}{R} \right)^n, \quad z \in D(0, R)$$

and

$$\left| \, f^{(n)}(0) \, \right| \leq \frac{M n!}{R^n}.$$

Moreover if we have equality in the first inequality for some $z \neq 0$, or if we have equality in the second inequality, then $f(z) = c z^n$ for some $c$ with $|\, c \,| = \frac{M}{R^n}$.

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*File ref: 611s2003˙asgn˙03.tex May 16, 2003

**Problem 3.** Suppose $f$ is analytic in $D(0,1)$, $f'(0) = 0$ and $|\, f(z) \,| \leq 1$ for each $z \in D(0,1)$. Show that

$$|\, f(z) - f(-z) \,| \leq 2 \,|\, z \,|^3, \quad z \in D(0,1).$$

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**Problem 4.** Suppose $f$ is analytic in $\Omega = \{\, z \in \mathbb{C} \;\mid\; |\, z \,| > R \,\}$, $|\, f(z) \,| \leq M$ for $z \in \Omega$ and

$$\lim_{|\, z \,| \to \infty} f(z) = 0.$$

Show that

$$|\, f(z) \,| \leq \frac{MR}{|\, z \,|}, \quad z \in \Omega.$$

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**Problem 5.** If we expand

$$\frac{1}{z^2 + z + 1}$$

in a power series about $z = 2$, what is the radius of convergence?

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**Problem 6.** Suppose $f$ is analytic in a neighborhood of the annulus $\{\, z \in \mathbb{C} \;\mid\; 2 \leq |\, z \,| \leq 3 \,\}$ and suppose

$$\max_{|\, z \,| = 2} |\, f(z) \,| \leq 16, \quad \text{and} \quad \max_{|\, z \,| = 3} |\, f(z) \,| \leq 54.$$

Show that

$$|\, f(z) \,| \leq 2 \,|\, z \,|^3, \quad 2 \leq |\, z \,| \leq 3.$$

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**Problem 7.** Let $f$ be an entire function and as usual let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

and

$$M(r) = \max_{|\, z \,| = r} |\, f(z) \,|\,.$$

Show that

$$\limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} = \limsup_{n \to \infty} \frac{n \log n}{- \log |\, c_n \,|}.$$

This common value is called the *order* of $f$.

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**Problem 8.** If $\rho > 0$ and $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is an entire function we define the $\rho$-type $\tau$ of $f$ by

$$\tau = \frac{1}{\rho \mathrm{e}} \limsup_{n \to \infty} n \,|\, c_n \,|^{\frac{\rho}{n}}\,.$$

Show that an entire function $f$ has exponential type if and only if it has order $\rho < 1$ or has order $\rho = 1$ and finite 1-type.

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**Problem 9.** Let $f$ be an entire function and let $M(r) = \max_{|\,z\,|=r} |\,f(z)\,|$. Let $0 \leq \rho < \infty$. If

$$\limsup_{r\to\infty} r^{-\rho} \log M(r) = \tau$$

where $0 < \tau < \infty$ then show $f$ has order $\rho$ and $\rho$-type $\tau$.

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**Problem 10.** Let $p(z) = a_0 + a_1 z + \cdots a_n z^n$ and let $M > 0$ and $R > 0$. Assume $|\,p(z)\,| \leq M$ for $|\,z\,| = R$. Show that

$$|\,p(z)\,| \leq M \left|\,\frac{z}{R}\,\right|^n, \quad \text{if } |\,z\,| \geq R.$$

**Hint:** The Cauchy inequalities imply $|\,a_n\,| \leq R^{-n} M$.

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**Problem 11.** Let $a_j > 0,\ j = 1, 2, 3, 4$. Let $f$ be continuous on $\overline{D(0,1)}$ and analytic on $D(0,1)$. Suppose $|\,f(z)\,| \leq a_j$ on the part of the unit circle in the $j^{th}$ quadrant. Show that

$$|\,f(0)\,| \leq (a_1 a_2 a_3 a_4)^{\frac{1}{4}}\,.$$

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**Problem 12.** Let $u, v \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. Let $f$ be an entire function and suppose

$$\begin{aligned} f(z+u) &= f(z) \\ f(z+v) &= f(z) \end{aligned}$$

for each $z \in \mathbb{C}$. Show that $f$ is constant.

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**Problem 13.** Let $p(z)$ be a polynomial with roots $a_1, a_2, \cdots, a_n$, repeated according to multiplicity. Then

$$\frac{p'(z)}{p(z)} = \sum_{k=1}^{n} \frac{1}{z - a_k}.$$

Hence prove Lucas's theorem: each root of $p'(z)$ lies in the convex hull of the roots of $p(z)$, that is, if $p'(z_0) = 0$ then there exist real numbers $\lambda_k \geq 0$ such that

$$\sum_{k=1}^{n} \lambda_k = 1 \quad \text{and} \quad z_0 = \sum_{k=1}^{n} \lambda_k a_k.$$

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**Problem 14.** Let $\mathbb{C}_\infty$ be the Riemann sphere, let $z_2, z_3, z_4$ be distinct points in $\mathbb{C}_\infty$ and let $w_2, w_3, w_4$ be distinct points in the Riemann sphere as well. By contemplating the equation

$$\frac{w - w_3}{w - w_4}\,\frac{w_2 - w_4}{w_2 - w_3} = \frac{z - z_3}{z - z_4}\,\frac{z_2 - z_4}{z_2 - z_3}$$

show there exists a unique Möbius transform $S$ such that $w_j = S(z_j),\ j = 2, 3, 4$. Let $R$ be the unique Möbius transform such that $R(z_2) = 1$, $R(z_3) = 0$ and $R(z_4) = \infty$. Then for any $z_1 \in \mathbb{C}_\infty$ we define the *cross-ratio*

$$(z_1, z_2, z_3, z_4) = R(z_1).$$

There are quite a few cases to consider to compute the cross-ratio. Show for example

$$\begin{aligned}(z_1, z_2, z_3, z_4) &= \frac{z_1 - z_3}{z_1 - z_4}\frac{z_2 - z_4}{z_2 - z_3} \quad \text{if } z_1, z_2, z_3, z_4 \in \mathbb{C}, z_1 \neq z_4 \\ (z_1, z_2, z_3, z_4) &= \infty \quad \text{if } z_1, z_2, z_3, z_4 \in \mathbb{C}, z_1 = z_4 \\ (z_1, \infty, z_3, z_4) &= \frac{z - z_3}{z - z_4} \quad \text{if } z_1, z_3, z_4 \in \mathbb{C}, z_1 \neq z_4\end{aligned}$$

Show if $T$ is any Möbius transform and $z_2, z_3, z_4$ are distinct points in $\mathbb{C}_\infty$ and $z_1 \in \mathbb{C}_\infty$ then

$$(T(z_1), T(z_2), T(z_3), T(z_4)) = (z_1, z_2, z_3, z_4)\,.$$

In particular if $T(z_j) = w_j$, $j = 2, 3, 4$ then

$$(T(z), w_2, w_3, w_4) = (z, z_2, z_3, z_4)\,,$$

a very handy formula. If $z \in \mathbb{C}_\infty$ show that

$$(z, 1, 0, \infty) = z$$

and therefore for any Möbius transform $S$ we have

$$S(z) = \left(z, S^{-1}(1), S^{-1}(0), S^{-1}(\infty)\right).$$

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**Problem 15.** If $\Gamma$ is a "circle" and $z_2, z_3, z_4$ are distinct points on $\Gamma$ then for each $z \in \mathbb{C}_\infty$ we define the conjugate $z^*_\Gamma$ by

$$z^*_\Gamma = S^{-1}\left(\overline{S(z)}\right)$$

where $S(z) = (z, z_2, z_3, z_4)$. Show that the conjugate is independent of the choice of $S$ (defined as above) and that $(z^*_\Gamma)^*_\Gamma = z$. If $\Gamma$ is the circle $|\,z - a\,| = R$, $z \neq a$, and $z \neq \infty$ then

$$(z^*_\Gamma - a)\,(\overline{z} - \overline{a}) = R^2, \quad a^*_\Gamma = \infty, \quad \infty^*_\Gamma = a.$$

**Remarks:** Points $a, b$ with $b = a^*_\Gamma$ are said to be symmetric with respect to $\Gamma$. For a line $L$ the points $a, b$ are symmetric with respect to $L$ if $L$ is the orthogonal bisector of the segment joining $a$ and $b$. For a circle, the geometric description of symmetry is a bit more complicated. You may enjoy finding it. Symmetry is preserved by Möbius transforms.

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**Problem 16.** Let $a \neq b$ be points in $\mathbb{C}$ and let $T$ be the Möbius transform

$$T(z) = \frac{z - a}{z - b}.$$

Show

$$\Gamma_\mu = T^{-1}\left(\{\, w \in \mathbb{C} \;\mid\; |\,w\,| = \mu \,\}\right)$$

is a family of circles such that $a$ and $b$ are $\Gamma_\mu$ symmetric for each $\mu$. These circles are the circles of Apollonius (with limit points $a$ and $b$). Show that $\Gamma_\mu$ has center $c$ and radius $r$ where

$$c = \frac{a - \mu^2 b}{1 - \mu^2}, \quad \text{and} \quad r = \frac{\mu\,|\,a - b\,|}{|\,1 - \mu^2\,|}.$$

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