Riemann Mapping Theorem and the Bergman Kernel

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These notes summarize the portion of our course that dealt with the Riemann conformal mapping theorem. I do not provide the detailed proofs that were given in class – just an overview – but perhaps I will have included some comments that were omitted or, worse, garbled in class.

1 Riemann's Conformal Mapping Theorem

Let Ω be a nonempty, connected and proper open subset of the complex plane. We say that Ω has the *square root property*, SRP, if for each function f analytic on Ω such that $f(z) \neq 0$ for each $z \in \Omega$, we have there exists g analytic on Ω such that $f(z) = g(z)^2$ for each $z \in \Omega$.

A simple calculation shows that if $f(z) \neq 0$ for each $z \in \Omega$ and f'/f has a primitive on Ω then there exists g analytic on Ω such that $f(z) = g(z)^2$ for each $z \in \Omega$. It follows that any proper nonempty connected simply connected open set has the SRP.

Let $D(a, r) = \{\, z \in \mathbb{C} \mid |\, z - a \,| < r \,\}$ and let $D = D(0, 1)$. Let $A(\Omega)$ be the space of all functions analytic on Ω.

Now Riemann's mapping theorem (1851) may be formulated as:

Theorem 1. *Let Ω have the SRP. If $a \in \Omega$ then there exists a unique $f \in A(\Omega)$ such that*

1. $f(a) = 0$ *and* $f'(a) > 0$,
2. f *is one-to-one,*
3. $f(\Omega) = D$.

Recall that a nonconstant analytic function is open. Thus the function f provided by Riemann's theorem is a homeomorphism. Even more, a one-to-one analytic function has non-vanishing derivative and so has analytic inverse.

Recall also that an analytic function with nonvanishing derivative preserves angles, and their orientation, so f is in fact *conformal.* Conversely a Frechét differentiable conformal map (preserving angles, including orientation) is analytic. So Riemann's mapping theorem is sometimes called the *the conformal mapping theorem.* We will refer to the unique function f given by theorem (1) as the *Riemann conformal map* for the pair (Ω, a).

Note if f is the Riemann conformal map for the pair (Ω, a) and g is be the Riemann conformal map for the pair (Ω, b) then

$$g(z) = \frac{|\, f'(b) \,|}{f'(b)} \, \frac{f(z) - f(b)}{1 - \overline{f(b)} f(z)}.$$

Since D is simply connected, and this property is preserved by homeomorphism, we have:

Corollary 2. *A nonempty proper connected open subset of $\mathbb{C}$ has the SRP if and only if it is simply connected.*

Corollary 3. *A nonempty proper connected open subset of $\mathbb{C}$ is conformally equivalent to D if and only if it is simply connected.*

Riemann's proof of the conformal mapping theorem required solving the Dirichlet problem. For this he relied on the Dirichlet principle. His proof therefore technically contained a gap.

The first complete technically correct proof of the conformal mapping theorem is generally accepted to be the one given by Koebe around 1908, though Osgood may have had one earlier. Other proofs were of course given – for example, by Schwarz around 1870. It may prove interesting to study the various proofs, their key ideas and their shortcomings!

1.1 The normal families argument

Koebe was the person who introduced the square root argument used in the proof common in modern textbooks – see for example [2], [1] or [3]. It owes much to Koebe, Fejer, F. Riesz, Carathéodory, Bierbach, Grönwall, Lindelöf, Montel and others. This proof is a compactness argument based on Montel's normal families theorem. Here is how it goes (in outline):

First of all, the lemma of Schwarz, allows us to show that all of the conformal automorphisms of the unit disk are Möbius transformations of the form

$$w = c\,\frac{z-a}{1-\bar{a}z}, \qquad |\,c\,| = 1.$$

From this fact the uniqueness is immediate. Next we introduce the family

$$\mathcal{F} = \{\, f \in A(\Omega) \;\mid\; f \text{ one-to-one}, f(a) = 0, f'(a) > 0, f(\Omega) \subseteq D \,\}\,.$$

This family is uniformly bounded, so normal (that is, relatively compact in the compact-open topology, or, if you prefer, in the topology of uniform convergence on compact subsets of Ω) by Montel's theorem. Koebe's square root construction is used to show that $\mathcal{F}$ is not empty.

The next step is to show that the closure of $\mathcal{F}$ is obtained by simply adjoining $\{0\}$. This part of the argument makes use of Hurwitz' theorem: a sequence of one-to-one analytic functions converging uniformly on compact sets has limit that is one-to-one or constant – Rouché's theorem, which counts solutions of an equation, could be used instead. It follows that $\mathcal{F} \cup \{0\}$ is compact and therefore the continuous map $g \longrightarrow g'(a)$ has a maximum at some point $f \in \mathcal{F}$. The continuity of the map is established by an argument that may be based on the Cauchy formula. By the maximum principle $f(\Omega) \subseteq D$. Finally Koebe's square root trick and some facts about Möbius transforms are used to show $f(\Omega) = D$.

1.2 The potential theory argument

Riemann's original potential theory approach based on the Dirichlet problem has been perfected over the years and can now also be used to give a complete proof of the conformal mapping theorem –

see [4]. Some of the contributors to this line of development were Perron, C. Neumann, Schwarz, Poincaré, Hilbert, Osgood, E.H. Taylor, and others.

A physical argument for using the Dirichlet principle to solve the Dirichlet problem was given as early as 1840 by Gauss. It was popularized in the late 1840's by Dirichlet, Kelvin, and others, and proofs were attempted. In 1870 Weierstrass gave an example to show that the Dirichlet principle is false, at least in the form usually given, and it fell into disrepute. In 1898 Hilbert resurrected it by rigorously proving it under suitable, but useful, hypotheses. Schwarz, C. Neumann and Poincaré developed new methods for solving the Dirichlet problem and considerable generality was achieved in the work of Perron.

The basic idea in this approach is that by solving the Dirichlet problem one can prove the existence of Green's function – but there is a simple relation between the Riemann map and Green's function (see below).

1.3 Another version of Riemann's theorem

Once we have a version of the Riemann mapping theorem it is not difficult to produce some useful variations. Here is an example:

Theorem 4. *Let Ω be a connected simply connected nonempty open subset of $\mathbb{C}$. Let $a \in \Omega$ and let*

$$\mathcal{F} = \{\, h \in A(\Omega) \;\mid\; h \textit{ one-to-one}, h(a) = 0, h'(a) = 1, |\,h\,| \textit{ bounded} \,\}.$$

Then there exists a unique $g \in \mathcal{F}$ such that

$$\sup_{z\in\Omega} |\,g(z)\,| = \inf_{h\in\mathcal{F}} \sup_{z\in\Omega} |\,h(z)\,|.$$

Moreover $g(\Omega)$ is a disk,

$$g(\Omega) = D(0, r_\Omega(a)).$$

The radius $r_\Omega(a)$ is called the *interior mapping radius* of Ω at a. It is not difficult to show that if g is any one-to-one analytic map of Ω onto $D(0,1)$ then

$$r_\Omega(z) = \frac{1 - |\,h(z)\,|^2}{|\,h'(z)\,|}.$$

Thus for the unit disk D we have $r_D(z) = 1 - |\,z\,|^2$ and for the upper half plane U we have $r_U(z) = 2\mathfrak{Re}(z)$.

Note if $h \in A(\Omega)$ is one-to-one, $h(a) = 0$, $h'(a) = 1$ and $h(\Omega) = D(0,R)$ then $R = r_\Omega(a)$ and h is the unique map given by the above theorem.

2 Remarks on the Dirichlet Problem

2.1 The Dirichlet problem

In the historical remarks above I probably lied to you through oversight or outright ignorance – but it was accidental. Now I am going to lie to you on purpose by over-simplifying.

Let Ω be an open subset of $\mathbb{C}$ and let q be a real function defined on the boundary $\partial\Omega$ of Ω. The *Dirichlet problem* is

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = q \text{ on } \partial\Omega. \end{cases} \tag{1}$$

Here Δ is the Laplace operator sometimes denoted ∇^2.

Harmonic functions, that is, solutions of Laplace's equation, $\Delta u = 0$, are invariant under conformal mapping. Thus in a sense Riemann's conformal mapping theorem reduces the Dirichlet problem for any connected simply connected proper open subset of $\mathbb{C}$ to the Dirichlet problem for the unit disk, where we have Poisson's formula available! Of course there is a catch – the conformal map can be very badly behaved at the boundary – in general we cannot obtain a solution in this manner. For sets with sufficiently nice boundaries though we can extend the conformal map to a homeomorphism of the closure of Ω and the closed unit disk and proceed along the lines indicated.

2.2 The Dirichlet principle

The *Dirichlet principle* states that the solution to the Dirichlet problem is the function u which among all functions v with $v = q$ on $\partial\Omega$ minimizes the *Dirichlet integral*

$$\iint_\Omega \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \, dxdy.$$

2.3 The Dirichlet integral

We can give a geometric "interpretation" of the Dirichlet integral (defined above). Let $g = u + iv$ be a one-to-one analytic function on Ω. Then in view of the Cauchy–Riemann equations the integrand in the Dirichlet integral for u may be written

$$\begin{aligned} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = J(u,v) \\ &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |\, g' \,|^2 \end{aligned}$$

where $J(u,v)$ is the Jacobian of the transformation $(x,y) \to (u,v)$. It follows that

$$\iint_\Omega \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \, dxdy = \iint_\Omega |\, g'(z) \,|^2 \, dxdy = \text{area of } g(\Omega).$$

2.4 Green's function

The *Green's function* of Ω is defined by

$$g(z,w) = -\log|\,z-w\,| + u_w(z)$$

where u_w is the solution to the Dirichlet problem

$$\begin{cases} \Delta u_w = 0 \text{ in } \Omega \\ u_w(z) = \log|\,z-w\,| \text{ for } z \in \Omega. \end{cases}$$

The logarithm occurs here because it is the fundamental solution of the Laplacian in dimension 2. In dimension 3 or higher the appropriate term is $|\,z-w\,|^{2-n}$.

Green's formula is that the solution to the Dirichlet problem 1 is given by

$$u(w) = -\frac{1}{2\pi}\int_{\partial\Omega} q(z)\frac{\partial g}{\partial n_z}(z,w)ds_z. \tag{2}$$

Here derivative is the normal derivative at $\partial\Omega$ of $g(z,w)$ with respect to z and ds_z denotes the element of arc length along $\partial\Omega$.

As an example, Green's function for the disk $D(0,R)$ is given by

$$g(z,w) = \frac{1}{2}\log\frac{R^2 - 2\mathfrak{Re}(z\overline{w}) + |\,z\,|^2\,|\,w\,|^2\,R^{-2}}{|\,z\,|^2 - 2\mathfrak{Re}(z\overline{w}) + |\,w\,|^2}.$$

In this case Green's formula just reduces to Poisson's formula for the disk.

2.5 Relation between Green's function and Riemann's conformal mapping

If Ω is a simply connected, connected proper nonempty open subset of $\mathbb{C}$ and for each $w \in \Omega$ we let $f_w\colon\Omega \to D$ be the unique analytic bijection with $f_w(w) = 0$ and $f'_w(w) > 0$ thne the Green's function is given by

$$g(z,w) = -\log|\,f_w(z)\,|\,. \tag{3}$$

Conversely if we know Green's function then equation (3) allows us to find the Riemann conformal map. If we locally choose a harmonic conjugate v for $z \to g(z,w)$ then $f_w(z) = \exp(-g(z,w) - \mathrm{i}v(z))$ is actually well-defined and has a removeable singularity at w. Then f_w is the Riemann map for the pair (Ω, w) (see [4]).

In summary we see if we can find the Riemann map then we can find Green's function and so solve the Dirichlet problem. Conversely if we can solve the Dirichlet problem then we can find Green's function and therefore we can find the Riemann conformal map.

2.6 Hilbert transform

Let's make a few more informal connections. Let q be a function on the circle ∂D. Define

$$h(z) = \frac{1}{2\pi} \int_{\partial D} \frac{w+z}{w-z} \frac{q(w)}{w} \, dw$$

where ∂D is parametrized in the usual way. If we let $u = \mathfrak{Re} h$ then a quick calculation shows

$$u(r\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} \, q(\mathrm{e}^{\mathrm{i}\phi}) \, d\phi.$$

This is the Poisson formula for the unit disk, and as mentioned above, is a special case of Green's formula. Now let $v = \mathfrak{Im} h$ so v is the harmonic conjugate of u which satisfies $v(0) = 0$. A calculation shows

$$v(r\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2} \, q(\mathrm{e}^{\mathrm{i}\theta}) d\phi.$$

If we formally let r tend to 1 we obtain a divergent integral which does however exist in the principal value sense:

$$v(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cot\left(\frac{\theta - phi}{2}\right) \, q(\mathrm{e}^{\mathrm{i}\phi}) d\phi.$$

This is the *Hilbert transform* – we view a function q as the boundary value of a harmonic function, then the bounday value of the harmonic conjugate is the Hilbert transform of q. The Hilbert transform is the prototype for the Calderón–Zygmund singular integral operators, which in turn led to many important applications in partial differential equations.

3 The Bergman Kernel

Let $f \in A(\Omega)$, $z \in \Omega$ and $0 < r < R$ where $R = \mathrm{dist}(z, \partial\Omega)$ is the distance from z to the boundary of Ω. Then by Cauchy's formula

$$f^{(n)}(z) = \frac{n!}{2\pi} \int_{-\pi}^{\pi} f(z + r\mathrm{e}^{\mathrm{i}\theta}) \, r^{-n} \mathrm{e}^{-\mathrm{i}n\theta} \, d\theta.$$

If we take the absolute value, multiply this equation by r^{n+1}, and then integrate r from 0 to R we obtain

$$\frac{R^{n+2}}{n+2} \left| \, f^{(n)}(z) \, \right| \le \frac{n!}{2\pi} \iint_{D(z,R)} | \, f(z) \, | \, dxdy.$$

If we estimate the right side by the Cauchy-Schwarz inequality we obtain

$$\left| \, f^{(n)}(z) \, \right| \le \frac{n+2}{2} \frac{n!}{\pi^{1/2} R^{n+1}} \left(\iint_{D(z,R)} | \, f(z) \, |^2 \, dxdy \right)^{1/2}.$$

If we define the $\mathcal{L}^2$ norm of f by

$$\|f\| = \left(\iint_{\Omega} | \, f(z) \, |^2 \, dxdy \right)^{1/2}$$

then we have shown

$$\left| \, f^{(n)}(z) \, \right| \le \frac{n! \, (n+2)}{2} \frac{\|f\|}{\pi^{1/2} \, (\mathrm{dist}(z, \partial\Omega))^{n+1}}.$$

One can obtain the same estimate, but with $(n+2)/2$ replaced by $\sqrt{n+1}$ by expanding f in a power series in $D(z,R)$ and then computing the $\mathcal{L}^2$ norm over $D(z,R)$.

The estimate above certainly implies that for each compact set $K \subseteq \Omega$ there is a constant $C_K > 0$ such that

$$\sup_{z\in K} |\, f(z)\,| \le C_K \, \|f\| \,.$$

It follows that the space

$$A^2(\Omega) = A(\Omega) \cap \mathcal{L}^2(\Omega)$$

is a closed subspace of $\mathcal{L}^2(\Omega)$. Thus $A^2(\Omega)$ is a separable Hilbert space.

The estimate also shows that for each $z \in \Omega$ the linear functional $f \to f(z)$ is continuous on $A^2(\Omega)$. By the Riesz representation theorem there exists a unique function $B_z \in A^2(\Omega)$ such that

$$f(z) = (\, f \;|\; B_z\,)\,, \quad \text{for each } f \in A^2(\Omega).$$

We define

$$B(z,w) = \overline{B_z(w)}.$$

The function B is called the Bergman kernel of Ω. It was introduced by Bergman in 1921. The definition shows

$$f(z) = \iint_\Omega B(z,w) f(w)\, dxdy, \quad f \in A^2(\Omega),$$

where $w = x + iy$. This property is known as the *reproducing property.*

Let $(\psi_n)_{n\ge 0}$ be an orthonormal Hilbert basis of $A^2(\Omega)$. Then we have the Fourier–Bessel series

$$B_z = \sum_{n=0}^{\infty} b_n(z)\psi_n$$

with convergence in $A^2(\Omega)$ for each $z \in \Omega$. Since

$$\psi_n(z) = (\, \psi_n \;|\; B_z\,) = \overline{b_n(z)}$$

we see that

$$\sum_{n=0}^{\infty} |\, \psi_n(z)\,|^2 = \|B_z\|$$

and

$$B(z,w) = \sum_{n=0}^{\infty} \psi_n(z)\overline{\psi_n(w)},$$

with, a priori, convergence in $\mathcal{L}^2(\Omega)$, for each $z \in \Omega$.

For any $f \in A^2(\Omega)$ we have the Fourier–Bessel series

$$f = \sum_{n=0}^{\infty} (\, f \;|\; \psi_n\,)\, \psi_n$$

which converges in $A^2(\Omega)$. Actually more is true:

Lemma 5. *If $f \in A^2(\Omega)$ then the Fourier–Bessel series of f converges to f uniformly on compact subsets of Ω.*

Proof. Indeed

$$\left| f(z) - \sum_{n=0}^{m} (\, f \mid \psi_n \,) \, \psi_n(z) \right| \leq \frac{\| f - \sum_{n=0}^{m} (\, f \mid \psi_n \,) \, \psi_n \|}{\pi^{1/2} \mathrm{dist}(z, \partial\Omega)}.$$

□

In particular the series

$$B_z = \sum_{n=0}^{\infty} b_n(z) \psi_n$$

converges uniformly on compact subsets of Ω for each $z \in \Omega$. Thus the series

$$B(z, w) = \sum_{n=0}^{\infty} \psi_n(z) \overline{\psi_n(w)}$$

converges at least pointwise on Ω. It follows that

$$B(z, z) = \sum_{n=0}^{\infty} |\, \psi_n(z) \,|^2 = \sum_{n=0}^{\infty} |\, b_n(z) \,|^2 = \|B_z\|^2 .$$

Now $B_z \in A^2(\Omega)$ implies

$$|\, B_z(w) \,| \leq \frac{\|B_z\|}{\pi^{1/2} \mathrm{dist}(w, \partial\Omega)}.$$

Thus

$$\|B_z\|^2 = B(z, z) = |\, B_z(z) \,| \leq \frac{\|B_z\|}{\pi^{1/2} \mathrm{dist}(w, \partial\Omega)}$$

which implies;

Theorem 6.

$$\sum_{n=0}^{\infty} |\, \psi_n(z) \,|^2 \leq \frac{1}{\pi \, dist(z, \partial\Omega)^2}.$$

By the Cauchy–Schwarz inequality we now deduce that the series

$$B(z, w) = \sum_{n=0}^{\infty} \psi_n(z) \overline{\psi_n(w)}$$

converges absolutely, and converges uniformly on compact subsets of $\Omega \times \Omega$.

For any $f \in A^2(\Omega)$ by Cauchy–Schwarz we have

$$\left(\sum_{n=0}^{\infty} |\, (\, f \mid \psi_n \,) \,| \, |\, \psi_n(z) \,| \right)^2 \leq \left(\sum_{n=0}^{\infty} |\, (\, f \mid \psi_n \,) \,|^2 \right) \left(\sum_{n=0}^{\infty} |\, \psi_n(z) \,|^2 \right) \leq \|f\|^2 \, \frac{1}{\pi \mathrm{dist}(z, \partial\Omega)}$$

which implies:

Corollary 7. *If $f \in A^2(\Omega)$ then the Fourier–Bessel series of f converges absolutely.*

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