Complex Analysis II – Mth 515

Archive – Winter 1994 Files

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This archive contains the assignments from Mth 515 Winter 1994. Some of the problems have appeared on qualifying exams. Those problems are labeled **QE** (term) and some of them have hints added.

Contents

1 Assignment 1

Problem 1. Recall a pole of order n is a zero of multiplicity $-n$. Show if f is analytic in the punctured disk $D^*(a,R)$, $n \in \mathbb{Z}$ and f has a zero of multiplicity n at a then the derivative f' has a zero of multiplicity $n-1$ at a provided $n \neq 0$. (Give a clean proof which doesn't rely on breaking the problem into cases.)

Problem 2. Let Ω be a connected open subset of $\mathbb{C}$ and let $\mathcal{M}(\Omega)$ be the set of meromorphic functions on Ω. Show that $\mathcal{M}(\Omega)$ *is* a field. Why does Ω have to be connected?

Problem 3. Let $R > 0$, $a \in \mathbb{C}$ and let $(a_n)_{n\geq 1}$ be a sequence of distinct points in $D^*(a,R)$ converging to the point a. Suppose that f is a function analytic in $\Omega = D^*(a,R) \sim \{a_1,\, a_2,\, \cdots\}$ and suppose that $f(\Omega)$ is not dense in $\mathbb{C}$. Show that there exists N such that $n \geq N$ implies f has a removable singularity at a_n. Moreover show that f has a worst a pole at a. **Hint:** Look at the proof of the Casorati-Weierstrass theorem.

Problem 4. Let f be analytic in the disk $D(a,R)$, let $0 \leq r < R$, and let

$$M(r) = \max_{|z-a|=r} |\, f(z)\,|\,.$$

Let

$$f(z) = \sum_{n=0}^{\infty} a_n \, (z-a)^n$$

be the power series expansion of f in $D(a, R)$. Prove that

$$\sum_{n=0}^{\infty} |\, a_n \,|^2 \; r^{2n} \leq M(r)^2, \quad 0 \leq r < R.$$

Hint: Show that the sum is given by $\frac{1}{2\pi} \int_0^{2\pi} \left|\, f(a + r\mathrm{e}^{\mathrm{i}\theta}) \,\right|^2 \, d\theta$.

Problem 5. In the notation of the previous problem, recall the Cauchy inequalities

$$|\, a_n \,| \leq \frac{M(r)}{r^n}, \; n \in \mathbb{Z}, \, n \geq 0, \, 0 \leq r < R.$$

Show if we have equality for *some* n and for *some* r then $f(z) = a_n \, (z-a)^n$.

Problem 6. Let f be analytic in the annulus $\Omega = \{\, z \in \mathbb{C} \;\mid\; R_1 < |\, z - a \,| < R_2 \,\}$ where $0 \leq R_1 < R_2 \leq \infty$. Prove the Cauchy inequalities

$$|\, a_n \,| \leq \frac{M(r)}{r^n}, \; n \in \mathbb{Z}, \, R_1 < r < R_2.$$

Here the a_n are the coefficients in the Laurent expansion of f, that is, $f(z) = \sum_{n=-\infty}^{\infty} a_n \, (z-a)^n$.

Problem 7. (Lagrange interpolation polynomial) Let f be analytic in an open set Ω. Let γ be a closed rectifiable curve null-homotopic in Ω. Let $z_0, \cdots, z_m$ be $m+1$ distinct points in $\Omega \sim \text{traj.}\gamma$ such that γ has winding number 1 about each z_n. Let

$$Q(z) = \prod_{n=0}^{m} (z - z_n)$$

and let

$$P(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{Q(w)} \, \frac{Q(w) - Q(z)}{w - z} \, dw.$$

Show that P is a polynomial of degree $\leq m$ and that

$$P(z_n) = f(z_n) \quad 0 \leq n \leq m.$$

2 Assignment 2

Problem 8. Let f be analytic in $D^*(0, R)$ for some $R > 0$. Suppose f is an even function and has a pole at the origin. Show that the pole must have even order. Show the residue at the origin must be 0. **Hint:** Try looking at the integral formula for the Laurent series coefficients.

Problem 9. Let Ω be an open subset of $\mathbb{C}$ and let f and g be analytic functions on Ω. Suppose g is not identically zero in any component of Ω. Suppose there is a locally bounded nonnegative function $\phi \colon \Omega \to \mathbb{R}$ such that

$$|\, f(z) \,| \leq \phi(z) \; |\, g(z) \,|, \quad \text{for each } z \in \Omega.$$

Prove that there exists a unique analytic function h on Ω such that

$$f(z) = h(z)\, g(z), \quad \text{for each } z \in \Omega.$$

Moreover $|\, h(z)\,| \leq \phi(z)$ for each $z \in \Omega$ if ϕ is continuous at each zero of g.. **Hint:** There is no problem defining h in $\Omega \sim Z_g$ where

$$Z_g = \{\, z \in \Omega \;\mid\; g(z) = 0 \,\}\,.$$

Now consider punctured neighborhoods of points of Z_g $\cdots$

Problem 10. (QE Spring 1988) Find all entire functions f satisfying $|\, f(z)\,| \leq |\, z\,|^5$ for each $z \in \mathbb{C}$. **Hint:** See problem above.

Problem 11. State the Cauchy estimates for the coefficients of the power series expansion of an analytic function. Then do the previous problem by using the Cauchy estimates.

Problem 12. (QE Spring 1988, Fall 1989) Prove Schwartz' lemma: if f is analytic in the open unit disk $D = D(0,1)$, $f(0) = 0$ and $|\, f(z)\,| \leq 1$ for $z \in D$ then $|\, f(z)\,| \leq |\, z\,|$ for each $z \in D$ and $|\, f'(0)\,| \leq 1$. If in addition $|\, f(z)\,| = |\, z\,|$ *for some* $z \in D \sim \{0\}$ or if $|\, f'(0)\,| = 1$ then there is a constant c of modulus 1 such that $f(z) = c\, z$ for each $z \in D$. **Hint:** Use the maximum modulus theorem (carefully) for both parts. Note we make no assumption about the behavior of f of the boundary of D so the proof is a bit more slippery than was required for the problem on QE Fall 1989.

Problem 13. (QE Fall 1991) If f is analytic in the unit disk $D = \{\, z \;\mid\; |\, z\,| < 1\,\}$, $|\, f(z)\,| \leq 1$ for each $z \in D$ and there exists an integer $m \geq 1$ such that $f^{(k)}(0) = 0$ for $0 \leq k < m$ prove that either (A) $|\, f(z)\,| < |\, z\,|^m$ for each $z \in D$ with $z \neq 0$ or (B) there exists a constant c with $|\, c\,| = 1$ such that $f(z) = c\, z^m$ for each $z \in D$. **Hint:** Here we have a variation on Schwartz' lemma. We note $f(z)\, z^{-m}$ has a removable singularity at the origin. Then we play delicately with the maximum principle $\cdots$

Problem 14. (QE Spring 1992) Let f be an entire function. Suppose $f(z)$ has a root at $z = \mathfrak{i}$ and at $z = -\mathrm{i}$. Let

$$M = \sup_{|z|=2} |\, f(z)\,|\,.$$

Prove

$$|\, f(z)\,| \leq \frac{M}{3}\, \left|\, z^2 + 1\,\right| \quad \text{if } |\, z\,| \leq 2.$$

Hint: Yet another variation on Schwarz' lemma, $f(z)\left(z^2 + 1\right)^{-1}$ has removable singularities and is easy to bound on the circle $\cdots$

Problem 15. Let f be an analytic function in an open set Ω and let $g(z) = \overline{f(z)}, \quad z \in \Omega$. Show if g is analytic in Ω then f is constant on each component of Ω. **Hint:** There's a one line proof using the open mapping theorem, a two line proof using the Cauchy–Riemann equations, and probably some multi–line proofs too.

3 Assignemnt 3

$$D(a,r) = \{\, z \in \mathbb{C} \mid |\, z - a \,| < r \,\} \text{ and } \partial D(a,r) = \{\, z \in \mathbb{C} \mid |\, z - a \,| = r \,\}$$

Problem 16. (QE Fall 1988) (A) State Rouché's theorem. (B) How many roots does $3\,z^3 - e^z$ have in the disk $|\,z\,| < 1$? **Hint:** Show $\left|\,\left(3\,z^3 - e^z\right) - 3\,z^3\,\right| < \left|\,3\,z^3\,\right|$ on $\partial D(0,1)$.

Problem 17. (QE Fall 1989) Let S be an open rectangle in the complex plane. Let $p(z)$ and $q(z)$ be polynomials given by $p(z) = \sum_{j=0}^{n} a_j\, z^j$ and $q(z) = \sum_{j=0}^{n} b_j\, z^j$. Suppose that $p(z)$ has no roots on the boundary of S and has k roots in S (counting each root according to its multiplicity). Prove: there exists a *positive* ϵ, such that if $\max_j |\, b_j - a_j\,| < \epsilon$ then $q(z)$ has exactly k roots in S (again counting multiplicity). **Hint:** Rouché's theorem.

Problem 18. (QE Spring 1990 - Modified) Let $f(w) = w^2 + w$. (A) Find analytic functions $w = g_j(z)$, $j = 1,2$ in $\Omega = \mathbb{C} \sim (-\infty, -1/4]$ such that $f(g_j(z)) = z$ for each $z \in \Omega$. Describe each g_j very carefully. (B) Find the range of g_j, $j = 1,2$. (C) Describe the image $f(\partial D(0,r))$ of the circle with center at the origin and radius r under f for $r = 1/2$, $r = 1$ and $r = 2$ and compute the winding numbers of each of these curves about the points z in their complements. (D) Describe the number of solutions in $D(0,r)$ of the equation $f(w) = z$ for each z in the complement of the image of the circle $\partial D(0,r)$ in terms of the winding numbers. Relate the solutions to the functions g_j.

Problem 19. (QE Spring 1990 - Modified) Let $q(z)$ be a polynomial of degree $\leq n$ and let $p(z) = q(z) + z^{n+1}$. Show that $|\,p(z_0)\,| \geq 1$ for some z_0 with $|\,z_0\,| = 1$. **Hint:** Rouché's theorem.

Problem 20. (QE Fall 1990) Let f be continuous in $\overline{D(0,1)}$ and analytic in $D(0,1)$. Assume that f has no zeros in $D(0,1)$ and $|\,f(z)\,| = 1$ for $|\,z\,| = 1$. Show that f is constant. **Hint:** Apply the maximum principle to $1/f$.

Problem 21. (QE Spring 1991 - Modified) Let f and g be continuous in $\overline{D(0,1)}$ and analytic in $D(0,1)$. Assume that f has no zeros on the *circle* $\partial D(0,1)$ and there is a constant C such that $|\,g(z)\,| \leq C\,|\,f(z)\,|$ for each $z \in D(0,1)$. Suppose in addition that $|\,g(z)\,| \leq |\,f(z)\,|$ for each $z \in \partial D(0,1)$. Prove that $|\,g(z)\,| \leq |\,f(z)\,|$ for each $z \in D(0,1)$.

Problem 22. (QE Spring 1991 - Modified) Let f be an analytic map of the unit disk $D(0,1)$ into itself. One of the consequences of Schwartz's lemma is

$$|\,f'(z)\,| \leq \frac{1 - |\,f(z)\,|^2}{1 - |\,z\,|^2} \quad \text{for each } z \in D(0,1).$$

(A) Prove this inequality. (B) Does there exist an analytic map f of the unit disk into itself with $f(1/2) = 3/4$ and $f'(1/2) = 2/3$?

Problem 23. (QE Fall 1992) Suppose f is analytic in the unit disk $D(0,1)$ and there is a constant $M > 0$ such that $|\,f(z)\,| \leq M$ in $D(0,1)$. Suppose that $\{z_k\}_{k=1}^{\infty}$ is a sequence in $D(0,1)$ such that $f(z_k) = 0$ for each k. (A) Prove that for each n

$$|\,f(z)\,| \leq M \prod_{k=1}^{n} \left| \frac{z - z_k}{1 - \overline{z_k}\, z} \right|.$$

(B) Now prove

$$\prod_{k=1}^{\infty} |\,z_k\,| \geq \frac{|\,f(0)\,|}{M}.$$

4 Assignemnt 4

Problem 24. Use the residue theorem to compute

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^n\theta \, d\theta$$

where $n \in \mathbb{Z}$ and $n \geq 0$.

Hint: $2\cos\theta = z + 1/z$ if $|z| = 1$,

Problem 25. (QE Spring 1991) (A) State the Riemann mapping theorem. **(B)** Finds an analytic function which maps the upper half plane one–to–one onto the unit disc.

Problem 26. (QE Fall 1991 and QE Fall 1993) State the Riemann mapping theorem. Explicitly construct a mapping of the type asserted to exist by the Riemann mapping theorem for the domain $\Omega = \{ z \in \mathbb{C} \mid \Re\mathfrak{e}\, z > 0, \Im\mathfrak{m}\, z > 0 \}$.

Problem 27. Let Ω be a connected open set in $\mathbb{C}$ and let f be an analytic function on Ω. Suppose $f(z) \neq 0$ for each $z \in \Omega$. Suppose in addition that

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for each closed polygonal path γ in Ω. **(A)** Show there exists a function F analytic on Ω such that

$$\mathrm{e}^{F(z)} = f(z), \qquad \text{for each } z \in \Omega.$$

(B) If Ω is simply connected then $\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$ for each closed rectifiable curve γ in Ω. Why?

Hint: Show the hypotheses imply f'/f has a primitive in Ω.

Problem 28. (QE Fall 1993) (A) If f and g are entire functions and have exactly the same zeros counted according to multiplicity then there is an entire function h such that

$$f(z) = g(z)\,\mathrm{e}^{h(z)}, \qquad \text{for each } z \in \mathbb{C}.$$

(B) Prove that the infinite product $\prod_{n=1}^\infty \left(1 - z^2/n^2\right)$ converges to an entire function. **(C)** Prove that

$$\sin \pi z = \mathrm{e}^{h(z)}\,\pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

for some entire function h. Actually $h = 0$ but you are not being asked to prove this fact.

Problem 29. Let

$$a_n = \begin{cases} 1 + \left(\frac{n}{2}+1\right)^{-1/2} & \text{if } n \text{ is even} \\ 1 - \left(\frac{n-1}{2}+1\right)^{-1/2} & \text{if } n \geq 3 \text{ is odd} \end{cases}$$

$$b_n = \begin{cases} 1 - \left(\frac{n}{2}+1\right)^{-1/2} & \text{if } n \text{ is even} \\ 1 + \left(\frac{n-1}{2}+1\right)^{-1/2} + \frac{\left(\frac{n-1}{2}+1\right)^{-1/2}+1}{(n-1)/2} & \text{if } n \geq 3 \text{ is odd} \end{cases}$$

(A) Show $\sum_{n=1}^\infty (1 - a_n)$ converges (conditionally) and $\prod_{n=1}^\infty a_n$ diverges (to 0). **(B)** Show $\sum_{n=1}^\infty (1 - b_n)$ diverges and $\prod_{n=1}^\infty b_n$ converges (conditionally). **(C)** State a related theorem which guarantees absolute convergence of $\prod_{n=1}^\infty z_n$.

5 Contact Information

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