Complex Analysis II – Mth 515

Archive – Spring 1997 Files

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This archive contains some problems from Mth 515 Spring 1997. I may have lost some of the original problems.

Contents

1 Problem Set

Problem 1. Let $f_z(w) = \frac{1}{(w-1)(w-1+z)}$. If $z \neq 0$ show that f_z has a simple pole at $w = 1$ and the residue at that pole is $\frac{1}{z}$. If $z = 0$ then f_z has a pole of order 2 at $w = 1$ and the residue is 0. We know the residue is given by the integral

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma f_z(w)\, dw$$

where γ is a sufficiently small positively oriented circle about $w = 1$. What happens to this integral as $z \to 0$? Why?

Problem 2. In the preceding problem, it is pretty clear what is going on, but sometimes it can be hard to see. Let $z \in \mathbb{C}$, $z \neq 1$ and define the function f_z by

$$f_z(w) = \frac{w + \mathrm{e}^{\frac{z}{2}\left(w - \frac{1}{w}\right)}}{w - \mathrm{e}^{\frac{z}{2}\left(w - \frac{1}{w}\right)}}.$$

This function comes up in the study of certain Kapteyn series, a study that ultimately derives from Kepler's problem – so it's hardly just pulled out of the hat. Show that f_z has a simple pole at $w = 1$ and that the residue at the pole is $\frac{2}{1-z}$. If $z = 1$ show that f_z has a pole of order 3 at $w = 1$ and the residue is $\frac{-27}{5}$.

Problem 3. The function h_z given by

$$h_z(w) = \mathrm{e}^{\frac{z}{2}\left(w - \frac{1}{w}\right)}$$

has an isolated singularity at $w = 0$ and is analytic in $\mathbb{C} \sim \{0\}$. Hence we have a LAURENT series

$$h_z(w) = \sum_{n=-\infty}^{n=\infty} \mathrm{J}_n(z)\, w^n,$$

with normal convergence in $\mathbb{C} \sim \{0\}$. Note

$$\mathrm{J}_{-n}(z) = \mathrm{J}_n(-z) = (-1)^n\, \mathrm{J}_n(z).$$

If $\gamma(t) = \mathrm{e}^{\mathrm{i}t}$, $0 \le t \le 2\pi$, then the formula for the coefficients of the Laurent series yields

$$\mathrm{J}_n(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma w^{-n-1} \mathrm{e}^{z(w-1/w)/2}\, dw.$$

It follows that J_n is an entire function. Show

$$\mathrm{J}_n(z) = \frac{1}{\pi} \int_0^\pi \cos\left(z \sin t - nt\right)\, dt.$$

The function J_n is called the BESSEL function of order n.

Problem 4. If $f : D(a,R) \to \mathbb{C}$ is analytic, there is $b \in D(a,R)$ such that $f(b) = 0$ and there is a constant $M > 0$ such that $f(z) \le M$ for each $z \in D(a,R)$ then

$$|\, f(z)\,| \;\le\; \frac{MR\,|\, z-b\,|}{\left|\, R^2 - \overline{(b-a)}(z-a)\,\right|} \tag{1}$$

$$\left|\, f'(b)\,\right| \;\le\; \frac{MR}{R^2 - |\, b-a\,|^2}. \tag{2}$$

If we have equality in (1) for some $z \in D^*(a,R)$ or equality in (2) then there exists $c \in \mathbb{C}$ with $|\, c\,| = 1$ such that

$$f(z) = Mc\,\frac{R(z-b)}{R^2 - \overline{(b-a)}(z-a)}.$$

Hint: Make a change of variable in SCHWARZ' lemma. Another approach is simply to reprove SCHWARZ' lemma in the case off-center root, by multiplying by suitable MÖBIUS transformation in place of the division by z. The second approach is the one to use in the case of several roots – see the next few problems.

Problem 5. Suppose $f : \overline{D(0,1)} \to \mathbb{C}$ is continuous, f is analytic in $D(0,1)$ and $|\, f(z)\,| \le M$ for $z \in D(0,1)$. Let $a_1, a_2, \cdots, a_m$ be roots of f in $D(0,1)$. Show that

$$|\, f(z)\,| \le M \prod_{k=1}^{m} \left|\, \frac{z-a_k}{1-\overline{a}_k z}\,\right|$$

for each $z \in \overline{D(0,1)}$. Moreover, if we have equality for some $z_0 \ne a_k$ then there is a constant c with $|\, c\,| = 1$ such that

$$f(z) = Mc \prod_{k=1}^{m} \frac{z-a_k}{1-\overline{a}_k z}$$

for each $z \in \overline{D(0,1)}$.

Hint: $\left| \frac{z-a_k}{1-\overline{a}_k z} \right| = 1$ if $|\, z \,| = 1$. Let

$$g(z) = f(z) \prod_{k=1}^{m} \frac{1-\overline{a}_k z}{z-a_k}$$

and use the maximum principle to estimate g.

Problem 6. Suppose $f : D(0,R) \to \mathbb{C}$ is analytic and $|\, f(z) \,| \le M$ for $z \in D(0,R)$. Let $a_1, a_2, \cdots, a_m$ be roots of f in $D(0,R)$. Show that

$$|\, f(z) \,| \le MR \prod_{k=1}^{m} \left| \frac{z-a_k}{R^2 - \overline{a}_k z} \right|$$

for each $z \in D(0,R)$. Moreover, if we have equality for some $z_0 \neq a_k$ then there is a constant c with $|\, c \,| = 1$ such that

$$f(z) = MR \prod_{k=1}^{m} \frac{z-a_k}{R^2 - \overline{a}_k z}$$

for each $z \in D(0,R)$.

Hint: If $0 < r < R$ use a change of variable to obtain the estimate

$$|\, f(z) \,| \le Mr \prod_{k=1}^{m} \left| \frac{z-a_k}{r^2 - \overline{a}_k z} \right|$$

in $\overline{D(0,r)}$.

Problem 7. If $f : D(0,1) \to \mathbb{C}$ is analytic, $f(-\frac{1}{2}) = f(\frac{1}{2}) = 0$, $f(0) = 1$ and $|\, f(z) \,| \le 4$ for each $z \in D(0,1)$ then

$$f(z) = 4\,\frac{1-4z^2}{4-z^2}.$$

Problem 8. Suppose we have real numbers $0 < r_1 < r_2 < r_3 < \cdots < r_n < R$ and real numbers $\lambda_k \ge 0$, $k = 1, 2, 3, \cdots, n$. Define

$$n(t) = \sum_{r_k < t} \lambda_k, \quad t \in [0, R]\,.$$

Show that

$$\int_0^R \frac{n(t)}{t}\, dt = \sum_{k=1}^{n} \log \frac{R}{r_k}.$$

Now let f be analytic in $D(a,R)$ and suppose $f(a) \neq 0$. Suppose f has a finite number of roots $a_1, a_2, \cdots, a_n$ in $D(a,R)$ (repeated according to multiplicity). Let $n(r)$ be the number of roots in $D(a,r)$. Show

$$\int_0^R \frac{n(t)}{t}\, dt = \sum_{k=1}^{n} \log \frac{R}{|\, a_k \,|}.$$

Problem 9. Prove the POISSON formula: if g is analytic in $D(0,R)$ and $u(z) = \mathfrak{Re}\, g(z)$ then

$$g(z) = \mathrm{i}\,\mathfrak{Im}\, g(0) + \frac{1}{2\pi}\int_0^{2\pi} \frac{r\mathrm{e}^{\mathrm{i}\theta}+z}{r\mathrm{e}^{\mathrm{i}\theta}-z}\, u(r\mathrm{e}^{\mathrm{i}\theta})\, d\theta$$

for $0 < r < R$ and $z \in D(0,r)$.

Hint: First show

$$\frac{r\mathrm{e}^{\mathrm{i}\theta}+z}{r\mathrm{e}^{\mathrm{i}\theta}-z} = 1 + 2\sum_{n=1}^{\infty} z^n r^{-n} \mathrm{e}^{-\mathrm{i}n\theta}$$

uniformly in real θ. Next note if $g(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$u(r\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{2}\sum_{n=0}^{\infty} r^n \left(a_n \mathrm{e}^{\mathrm{i}n\theta} + \overline{a}_n \mathrm{e}^{-\mathrm{i}n\theta}\right)$$

for $0 \le r < R$, uniformly in real θ.

Problem 10. If F is analytic in $D(0,R)$ and has no roots in $D(0,R)$ then for each $0 \le r < R$ and $z \in D(0,r)$ we have

$$\log|\, F(z)\,| = \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{Re}\left(\frac{r\mathrm{e}^{\mathrm{i}\theta}+z}{r\mathrm{e}^{\mathrm{i}\theta}-z}\right) \log\left|\, F(r\mathrm{e}^{\mathrm{i}\theta})\,\right|\, d\theta.$$

Hint: There is a branch of $\log F(z)$ in $D(0,R)$.

Problem 11. Prove the POISSON–JENSEN formula. Let f be analytic in $D(0,R)$, let $0 < r < R$ and let $a_1, a_2, \cdots, a_m$ be the roots of f in $D(0,r)$. If $z \in D(0,r)$ and $f(z) \neq 0$ then

$$\log|\, f(z)\,| + \sum_{k=1}^{m} \log\left|\, \frac{r^2 - \overline{a}_k z}{r(z-a_k)}\,\right| = \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{Re}\left(\frac{r\mathrm{e}^{\mathrm{i}\theta}+z}{r\mathrm{e}^{\mathrm{i}\theta}-z}\right) \log\left|\, f(r\mathrm{e}^{\mathrm{i}\theta})\,\right|\, d\theta.$$

Note that f is permitted to have roots on $\partial D(0,r)$.

Hint: First assume that f has no roots on $\partial D(0,r)$ and let

$$F(z) = f(z)\prod_{k=1}^{m} \frac{r^2 - \overline{a}_k z}{r(z-a_k)}$$

and use the previous exercise. Now if f has roots on $\partial D(0,r)$ then choose $0 < s < r$ such that $a_k \in D(0,s)$ for each $k = 1, \cdots, n$. Then f has no roots on $\partial D(0,t)$ for $s < t < r$. Now investigate the integral as $t \to r$.

Problem 12. Deduce JENSEN's equality (1899) from the previous exercise. Suppose f is analytic in $D(0,R)$. For each $0 < t < R$ let $n(t)$ be the number of roots (counted according to multiplicity) of f in $D(0,t)$. If $f(0) \neq 0$ and $0 < r < R$ show

$$\log|\, f(0)\,| + \int_0^r \frac{n(t)}{t}\, dt = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\, f(r\mathrm{e}^{\mathrm{i}\theta})\,\right|\, d\theta.$$

(Note we could just as well take $n(t)$ to be the number of roots in $\overline{D(0,t)}$. It does not make any difference.)

The investigation of the integral in the proof of the POISSON-JENSEN formula above may appear a bit fearsome. Here then is a slightly different approach to the proof of JENSEN's equality. Suppose f is analytic in $D(0,R)$, $0 < r < R$ and f has no roots in $D(0,r)$. Let

$$re^{i\theta_1}, \cdots, re^{i\theta_n}$$

be the roots of f on $\partial D(0,r)$. Then there is an analytic function $g\colon D(0,R) \to \mathbb{C}$ such that

$$f(z) = g(z) \prod_{j=1}^{n} \left(z - re^{i\theta_j} \right).$$

For some $\epsilon > 0$, g has no roots in $D(0, r+\epsilon)$. It follows there is a branch of $\log g(z)$ in $D(0, r+\epsilon)$. Taking the real part in CAUCHY's integral theorem we obtain

$$\log |\, g(0) \,| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| g\left(re^{i\theta}\right) \right| d\theta.$$

Now

$$\log |\, f(0) \,| = \log |\, g(0) \,| + n \log r$$

and

$$\log \left| f\left(re^{i\theta}\right) \right| = \log \left| g\left(re^{i\theta}\right) \right| + n \log r + \sum_{j=1}^{n} \log \left| e^{i\theta} - e^{i\theta_j} \right|.$$

But

$$\begin{aligned} \int_0^{2\pi} \log \left| e^{i\theta} - e^{i\theta_j} \right| d\theta &= \int_0^{2\pi} \log \left| e^{i\theta} - 1 \right| d\theta \\ &= 2\pi \log 2 + 2 \int_0^{\pi} \log(\sin\theta)\, d\theta = 0. \end{aligned}$$

It now follows that

$$\log |\, f(0) \,| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f\left(re^{i\theta}\right) \right| d\theta.$$

This is JENSEN's equality for the case that f has no roots in $D(0,r)$. The general case we now handle by using MÖBIUS transforms as above (we do not use them as transforms – we simply multiply by them viewed as complex functions). This version of the proof may be due to BACKLUND (1918). It hinges on the following result:

Problem 13. Show

$$\int_0^{\pi} \log(\sin x)\, dx = -\pi \log 2.$$

Note the integral is actually absolutely convergent.

Hint: Integrate $\log\left(1 - e^{2iz}\right)$ along the rectangular contour with height h and base on the interval $[0, \pi]$, but with the lower two corners excised. Then let $h \to \infty$ and let the excisions get small. (It is a tricky calculation).

2 Contact Information

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