Complex Analysis I – Mth 514

Archive – Winter1997 Files

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This archive contains the assignments from Mth 514 Winter 1997.

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1 Assignment 1

Problem 1. A Möbius transform $T \neq I$ has ∞ as a fixed point if and only if there are complex numbers $a \neq 0$ and b (and $b \neq 0$ if $a = 1$) such that $T(z) = az + b$ for each $z \in \mathbb{C}_\infty$.

Problem 2. A Möbius transform $T \neq I$ has ∞ as its only fixed point if and only if there is a complex number $b \neq 0$ such that $T(z) = z + b$ for each $z \in \mathbb{C}_\infty$.

Problem 3. A Möbius transform $T \neq I$ has 0 as a fixed point if and only if there are complex numbers c and $d \neq 0$ (and $d \neq 1$ if $c = 0$) such that $T(z) = \dfrac{z}{cz + d}$ for each $z \in \mathbb{C}_\infty$.

Problem 4. A Möbius transform $T \neq I$ has 0 as its only fixed point if and only if there is a complex numbers $c \neq 0$ such that $T(z) = \dfrac{z}{cz + 1}$ for each $z \in \mathbb{C}_\infty$.

Problem 5. A Möbius transform $T \neq I$ has 0 and ∞ as fixed points if and only if there is a complex number $a \neq 0$ with $a \neq 1$ such that $T(z) = az$ for each $z \in \mathbb{C}_\infty$.

Problem 6. A Möbius transform $T \neq I$ has two distinct fixed points if and only if there exists a Möbius transform S and a complex number $a \neq 0$ with $a \neq 1$ such that $S^{-1}TS(z) = az$ for each $z \in \mathbb{C}_\infty$.

Problem 7. A Möbius transform $T \neq I$ has only one fixed point if and only if there exists a Möbius transform S such that $S^{-1}TS(z) = z + 1$ for each $z \in \mathbb{C}_\infty$. (Hint: If $R(z) = z + b$, $b \neq 0$ and $S(z) = bz$ then $S^{-1}RS(z) = z + 1$.)

Problem 8. If T is a Möbius transform then

$$T(z) = \big(z, T^{-1}(1), T^{-1}(0), T^{-1}(\infty)\big)$$

(cross–ratio) for each $z \in \mathbb{C}_\infty$.

Problem 9. If

$$T(z) = \frac{z-1}{z+1}$$

then T maps the right half plane onto the unit disk. Hint: Consider the images of 1, 0 and -1 and exploit symmetry.

Problem 10. (Circles of Apollonius) If $a \neq b$ are complex numbers and $\mu > 0$ then the equation

$$\left| \frac{z-a}{z-b} \right| = \mu$$

is the equation of a "circle" Γ_μ — it is a line if and only if $\mu = 1$. The points a and b are Γ_μ–symmetric for each $\mu > 0$. Conversely if Γ is any circle such that a and b are Γ–symmetric then $\Gamma = \Gamma_\mu$ for some μ. Hint: Think about the Möbius transform $S(z) = (z-a)/(z-b)$ and symmetry.

Problem 11. If S and T are Möbius transforms with the same fixed points then $ST = TS$. Hint: Consider the transforms $R^{-1}SR$ and $R^{-1}TR$ for a suitable Möbius transform R.

Problem 12. If S and T are Möbius transforms and $ST = TS$ and one of them has only one fixed point then the other has only one fixed point and S and T have the same fixed point.

Problem 13. If S and T are Möbius transforms and $ST = TS$ and one of them has two fixed points then the other has two fixed points. Remark: Moreover one can show they have the same fixed points or each one permutes the fixed points of the other one.

Problem 14. If T is a Möbius transform with fixed points $a \neq b$ then there is a complex number $\lambda \neq 0$ such that

$$\frac{T^n(z) - a}{T^n(z) - b} = \lambda^n \, \frac{z-a}{z-b}$$

for each $z \in \mathbb{C}_\infty$ and each integer $n \geq 1$. Hint: Consider the cross–ratio $(T(z), \infty, a, b)$ to get the case $n = 1$. Then choose R so $RTR^{-1}(w) = \lambda w$.

Problem 15. Let Γ be the circle with center on the real line at $c > 0$ and with radius $0 < \rho < c$. Let Γ' the the circle with the same radius but with center at $-c$. Find a Möbius transform which maps these circles to concentric circles with center at the origin. Hint: If $a = \sqrt{c^2 - \rho^2}$ then a and $-a$ are symmetric with respect to both circles. Now choose T with $T(-a) = \infty$, $T(0) = -1$ and $T(a) = 0$.

Problem 16. For each integer $n \geq 2$ find a Möbius transform $T \neq I$ such that $T(-1) = -1$, $T(1) = 1$ and $T^n = I$.

Problem 17. Let $0 \le t_n$ and let

$$0 < \alpha \le \liminf_{n\to\infty} \frac{t_n}{\log n} \quad \text{and} \quad \limsup_{n\to\infty} \frac{t_n}{\log n} \le \beta \le +\infty.$$

Then the series $\sum_{n=0}^{\infty} x^{t_n}$ converges for $0 \le x < \exp(-1/\alpha)$ and diverges for $x > \exp(-1/\beta)$.

In particular if $\liminf_{n\to\infty} \frac{t_n}{\log n} = \infty$ then the series $\sum_{n=0}^{\infty} x^{t_n}$ converges for $0 \le x < 1$ and diverges for $x > 1$.

Problem 18. If $\left| \sum_{k=0}^{n} b_k \right| \le M < \infty$ for each n, if $\sum_{n=0}^{\infty} |\, a_n - a_{n+1} \,|$ converges and if $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=0}^{\infty} a_n b_n$ converges.

Hint: Let $B_m = \sum_{k=0}^{m} b_k$. Then

$$\sum_{n=0}^{m} a_n b_n = \sum_{k=0}^{m-1} (a_k - a_{k+1})\, B_k + a_m B_m,$$

(partial summation).

Problem 19. Suppose $a_n \ge a_{n+1} \ge 0$ for each n. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k}$$

converges. Hint: Let A_n be partial sums of the first series and let B_k be partial sums of the second series. Show if $n < 2^k$ then $A_n \le B_k$ and if $n > 2^k$ then $2A_n \ge B_k$.

Problem 20. Use the result of the previous problem to obtain the usual convergence criteria for the series $\sum n^{-p}$. (Do not use the integral test.)

Problem 21. Let $a \in \mathbb{C}$. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} a^{n^2} z^n.$$

Problem 22. Let $a \in \mathbb{C}$. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \binom{2n}{n} z^n.$$

Consider two series $\sum a_n$ and $\sum b_n$. The CAUCHY product is the series $\sum c_n$ where $c_n = \sum_{k=0}^{m} a_k\, b_{n-k}$.

Problem 23. Prove MERTENS' theorem: if $A = \sum_{n=0}^{\infty} a_n$ is absolutely convergent and $B = \sum_{n=0}^{\infty} b_n$ is convergent, then the CAUCHY product converges to AB.

Hint: Let A_n, B_n and C_n be the appropriate partial sums. Show

$$AB - C_n = (A - A_n)\,B + \sum_{k=0}^{n} a_k\,(B - B_{n-k})$$

and use this to estimate the last term.

2 Assignment 2

Problem 24. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where the power series has radius of convergence $R > 0$. Let $0 \le r < R$. Show that

$$\int_0^{2\pi} \left|\, f\left(r\mathrm{e}^{\mathrm{i}\theta}\right) \right|^2 d\theta = 2\pi \sum_{n=0}^{\infty} |\, a_n\,|^2\, r^{2n}.$$

Conclude if $M(r) = \sup_{|\,z\,|=r} |\, f(z)\,|$ then

$$|\, a_n\,| \le \frac{M(r)}{r^n}.$$

(These inequalities are know as the CAUCHY estimates. See problem 14 below for another version and proof.)

Problem 25. Show that x^n converges on $[0,1]$, but not uniformly. Then show $x^n(1-x) \to 0$ uniformly on $[0,1]$ as $n \to \infty$.

Problem 26. Let $\gamma\colon [a,b] \to \mathbb{C}$ be of bounded variation. For each integer $n \ge 0$ let $f_n\colon [a,b] \to \mathbb{C}$ be RIEMANN–STIELTJES integrable with respect to $d\gamma$. Assume that $f_n \to f$ *uniformly on* $[a,b]$ as $n \to \infty$. Show that f is integrable with respect to $d\gamma$ and that

$$\lim_{n\to\infty} \int_a^b f_n(t)\, d\gamma(t) = \int_a^b f(t)\, d\gamma(t).$$

Notice we do not assume that the f_n are continuous so the integrability $d\gamma$ of f will require proof.

Problem 27. Prove the FUBINI–TONELLI theorem for the RIEMANN–STIELTJES integral. That is, show if $\phi\colon [a,b] \times [c,d] \to \mathbb{C}$ is continuous and $\gamma\colon [a,b] \to \mathbb{C}$ and $\beta\colon [c,d] \to \mathbb{C}$ are of bounded variation then

$$g(t) = \int_a^b \phi(s,t)\, d\gamma(s) \quad \text{and} \quad h(s) = \int_c^d \phi(s,t)\, d\beta(t)$$

define continuous functions $g\colon[c,d]\to\mathbb{C}$ and $h\colon[a,b]\to\mathbb{C}$. Moreover

$$\int_a^b h(s)\,d\gamma(s) = \int_c^d g(t)\,d\beta(t).$$

Problem 28. Use the FUBINI–TONELLI theorem and the fundamental theorem of calculus to give a careful proof of the theorem of LEIBNITZ: Let $\phi\colon[a,b]\times[c,c]\to\mathbb{C}$ be continuous and have continuous partial derivative $\dfrac{\partial\phi}{\partial t}\colon[a,b]\times[c,d]\to\mathbb{C}$. Let $\gamma\colon[a,b]\to\mathbb{C}$ be of bounded variation. If

$$g(t) = \int_a^b \phi(s,t)\,d\gamma(s)$$

the g is continuously differentiable on $[c,d]$ and

$$g'(t) = \int_a^b \frac{\partial\phi}{\partial t}(s,t)\,d\gamma(s).$$

Problem 29. Let $\gamma\colon[0,1]\to\mathbb{C}$ be defined by $\gamma(t)=1-t$. Show that

$$\int_\gamma \frac{dw}{w-z} = \log\left(\frac{z}{z-1}\right) \qquad \text{for each } z\notin\mathbb{C}\sim[0,1]\,.$$

Here log denotes the principal branch of the logarithm. **Hint:** $T(z)=z/(z-1)$ defines a MÖBIUS transform T with $T([0,1))=(-\infty,0]$.

Problem 30. If f is analytic in $D(a,R)$, $0<r<R$ and $\gamma(s)=a+r\mathrm{e}^{\mathrm{i}s}$, $0\le s\le 2\pi$, then

$$\frac{1}{2\pi\mathrm{i}}\int_\gamma \overline{f(z)}\,dz = r^2\,\overline{f'(a)},$$

where the over–bar denotes conjugation.

So far we have only proved a preliminary version of the CAUCHY theorems:

Theorem 2.1. *Let Ω be an open set in $\mathbb{C}$ and let $f:\Omega\to\mathbb{C}$ be analytic. Then f is infinitely often complex differentiable. If Ω_1 is a convex open subset of Ω and if $\gamma\colon[a,b]\to\Omega_1$ is a closed rectifiable curve then*

Part (a): CAUCHY Integral Formula:

$$\nu(\gamma,z)f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,dw$$

for each $z\in\mathbb{C}\sim\operatorname{traj}\gamma$ (makes sense since $\nu(\gamma,z)=0$ for each $z\notin\Omega_1$ by convexity of Ω_1).

Part (b): CAUCHY Theorem:

$$\int_\gamma f(w)\,dw = 0.$$

We will strengthen this theorem considerably, but even in its present from it is quite useful. We will use it to prove some important results. You can also use it to evaluate some integrals.

Problem 31. Evaluate
$$\int_\gamma \frac{e^{2w}}{(w-i)^5}\,dw$$
where γ is the circle $|\,w\,| = 2$ traversed once in the positive direction.

Problem 32. Evaluate
$$\int_\gamma \frac{1}{w^2+1}\,dw$$
where γ is the circle $|\,w\,| = 2$ traversed once in the positive direction.

Problem 33. Evaluate
$$\int_\gamma \frac{\sin w}{w^2}\,dw$$
where γ is the circle $|\,w\,| = 1$ traversed once in the positive direction.

Problem 34. Let $p(z)$ be a polynomial in z of degree m with complex coefficients. Let $f : \Omega \to \mathbb{C}$ be an analytic function where Ω is a convex open subset of $\mathbb{C}$. Let $a_1, a_2, \cdots, a_m$ be the roots of $p(z)$ repeated according to multiplicity. Let $\gamma : [a, b] \to \Omega \sim \{\, a_1, a_2, \cdots, a_m \,\}$ be a closed rectifiable curve. Show that
$$\frac{1}{2\pi i}\int_\gamma \frac{p'(z)f(z)}{p(z)}\,dz = \sum_{k=1}^{m} \nu(\gamma, a_k) f(a_k).$$
(Stronger versions of the CAUCHY theorems will yield stronger versions of this result.)

Problem 35. Show
$$\int_0^{2\pi} \frac{e^{it}}{e^{it}-z}\,dt = \begin{cases} 2\pi \text{ if } |\,z\,| < 1 \\ 0 \text{ if } |\,z\,| > 1. \end{cases}$$
Hint: Show the integral is equal to a suitable contour integral.

Problem 36. Let $p(z)$ be a polynomial of degree n. Let $a_1, a_2, \cdots, a_n$ be the roots of $p(z)$ repeated according to multiplicity. Show if $r > 0$ is large enough that all the roots lie in $D(0, r)$ then
$$\left|\sum_{k=1}^{n} a_k^m\right| \le \sup_{|\,z\,|=r} \left|\frac{p'(z)}{p(z)}\right| r^{m+1}$$
for each integer $m \ge 0$.

Problem 37. Let f be analytic in $D(0, R)$ for some $R > 0$. For each $0 \le r < R$ define
$$M(r) = \sup_{|\,z\,|=r} |\,f(z)\,|\,.$$
Show if $|\,z\,| < r < R$ then
$$|\,f(z)\,| \le \frac{M(r)r}{(r - |\,z\,|)}.$$
More generally
$$\left|\,f^{(n)}(z)\,\right| \le \frac{n!M(r)r}{(r - |\,z\,|)^{n+1}}.$$

Problem 38. Prove the theorem of LIOUVILLE: If f is a bounded entire function then f is constant. **Hint:** Use the result of the previous exercise with $n = 1$.

3 Assignment 3

Problem 39. Let Ω be an open set in $\mathbb{C}$ and let f be an analytic function on Ω. Let $z_0, z_1, \cdots, z_n \in \Omega$. Note these points are not assumed to be distinct. Choose a rectifiable curve γ in $\Omega \sim \{ z_0, \cdots, z_n \}$ such that

$$\begin{aligned} \nu(\gamma, z_j) &= 1 \quad j = 0, 1, \cdots, n \\ \nu(\gamma, z) &= 0 \quad \text{for each } z \notin \Omega. \end{aligned}$$

Then define the n^{th} Newton divided difference

$$f\,[z_0, z_1, \cdots, z_n] = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(w)}{(w - z_0)(w - z_1)\cdots(w - z_n)}\, dw.$$

Prove that $f\,[z_0, \cdots, z_n]$ is independent of the choice of γ. **Hint:** Show

$$f\,[z_0, z_1, \cdots, z_n] = \frac{f\,[z_1, \cdots, z_n] - f\,[z_0, \cdots, z_{n-1}]}{z_n - z_0}$$

if $z_n \neq z_0$ and

$$f\,[z_0, z_0, \cdots, z_0] = \frac{1}{n!} f^{(n)}(z_0).$$

Then use symmetry and induction.

Problem 40. Let Ω be an open set in $\mathbb{C}$ and let f be an analytic function on Ω. Let $z_0, z_1, \cdots, z_n \in \Omega$. Note these points are not assumed to be distinct. Show

$$g(z) = f\,[z_0, z_1, \cdots, z_n, z]$$

defines a function g analytic in Ω. **Hint:** By the previous problem you can use convenient choices for γ.

Problem 41. Let Ω be an open set in $\mathbb{C}$ and let f be an analytic function on Ω. Let $a \in \Omega$ and define

$$h(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a}, & z \neq a \\ f'(a), & z = a \end{cases}$$

Show that h is analytic on Ω. Do this problem two ways - once using the result of problem 2 above and once using power series (but be careful).

Problem 42. Let Ω be an open set in $\mathbb{C}$ and let f be an analytic function on Ω. Let $a \in \Omega$. Show that $f(a) = 0$ if and only if there exists a function h analytic on Ω such that

$$f(z) = (z - a)h(z), \quad z \in \Omega.$$

Do this problem two ways - once using the result of problem 3 above and once using power series (but be careful).

Problem 43. Let Ω be an open set in $\mathbb{C}$ and let f be an analytic function on Ω. Let $a \in \Omega$. If $0 < r < \operatorname{dist}(a, \partial\Omega)$ show that

$$f(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a + \mathrm{e}^{\mathrm{i}\theta})\, d\theta.$$

Multiply by r and integrate to deduce if $0 < R < \operatorname{dist}(a, \partial\Omega)$ then

$$f(a) = \frac{1}{\pi R^2} \iint\limits_{D(a,R)} f(x + iy)\, dxdy.$$

Conclude if $1 \le p$, $f \in \mathcal{L}^p(\Omega)$ and f is analytic on Ω then

$$|\, f(z) \,| \le \pi^{-1/p} \, \|f\|_p \operatorname{dist}(z, \partial\Omega)^{-2/p}.$$

Problem 44. Let Ω be an open subset of $\mathbb{C}$. Call a compact subset K of Ω *small in* Ω if K is contained in an open disk contained in Ω. Show that each compact subset of Ω is a finite union of small compact subsets of Ω.

Problem 45. Let Ω be an open subset of $\mathbb{C}$. Let (f_n) be a sequence of analytic functions on Ω and suppose $f_n \to f$ uniformly on each circle contained in Ω. Show that f is analytic and $f_n \to f$ uniformly on each compact subset of Ω. **Hint:** Cauchy integral theorem.

Problem 46. Let $\mathcal{H}(\Omega)$ denote the space of functions analytic on Ω. Let $p \ge 1$. Show if (f_n) is a sequence in $\mathcal{H}(\Omega) \cap \mathcal{L}^p(\Omega)$ and $f_n \to f$ in $\mathcal{L}^p(\Omega)$ then f is analytic and $f_n \to f$ uniformly on compact subsets of Ω. In particular $\mathcal{H}(\Omega) \cap \mathcal{L}^p(\Omega)$ is a closed subspace of $\mathcal{L}^p(\Omega)$.

Problem 47. If $f \in \mathcal{L}^p(\mathbb{C})$ is analytic then $f = 0$.

Problem 48. Let f be an entire function. Suppose there exist a complex number α and real numbers K and m such that

$$\mathfrak{Re}\,(\alpha f(z)) \le K + m \log(1 + |\, z \,|)$$

for each $z \in \mathbb{C}$. Show that f is constant. **Hint:** The Fundamental Theorem of Algebra plays a role.

Problem 49. Let Ω and Ω' be open subsets of $\mathbb{C}$. Let $f : \Omega \to \Omega'$ and $h : \Omega' \to \mathbb{C}$ be analytic functions. Let γ be a rectifiable curve in Ω. Prove that $f \circ \gamma$ is a rectifiable curve in Ω'. Now use the fact that h *locally* has a primitive to deduce in one or two lines that

$$\frac{1}{2\pi i} \int_{f\circ\gamma} h(w)\, dw = \frac{1}{2\pi i} \int_\gamma h(f(z)) f'(z)\, dz.$$

(This result is true in more generality, but then we have to work harder. Try it.)

Problem 50. Let Ω be an open subset of $\mathbb{C}$ and let f be a function analytic on Ω. Let γ be a closed rectifiable curve in Ω. Let $b \in \mathbb{C}$ and suppose $f(z) \ne b$ for each $z \in \operatorname{traj}\gamma$. Prove

$$\nu(f \circ \gamma, b) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - b}\, dz.$$

Problem 51. Let $p(z)$ be a polynomial of degree n with roots $a_1, \cdots, a_n$ repeated according to multiplicity. Let γ be a closed rectifiable curve in $\mathbb{C} \sim \{\, a_1, \cdots, a_n \,\}$. Show that

$$\sum_{k=1}^{n} \nu(\gamma, a_j) = \nu(p \circ \gamma, 0).$$

(This fact is the Principle of the Argument. With an additional hypothesis on γ it is true for analytic functions. We will prove it later.)

4 Assignment 4

Problem 52. Use the theorem on term–by–term differentiation of convergent power series to sum the series

$$\sum_{n=1}^{\infty} n^2 z^n.$$

Problem 53. Prove the DIRAC formula: if ϕ is a continuously differentiable function on $\mathbb{R}$ and ϕ vanishes outside of a compact set then

$$\lim_{y \downarrow 0} \int_{-\infty}^{\infty} \frac{\phi(x)}{x + iy}\, dx = -i\pi\phi(0) + \lim_{\epsilon \downarrow 0} \int_{|x| > \epsilon} \frac{\phi(x)}{x}\, dx.$$

Hint: Define $\log(z) = \log(|z|) + i \arg(z)$ where $-\pi/2 < \arg(z) < 3\pi/2$ and note if $y > 0$ then

$$\int_{-\infty}^{\infty} \frac{\phi(x)}{x + iy}\, dx = -\int_{-\infty}^{\infty} \log(x + iy)\phi'(x)\, dx.$$

Problem 54. Let γ be a circle of radius r parametrized in the usual way and suppose that γ does not pass through any of the roots of $z^3 + 1$. Find all possible values of

$$\int_\gamma \left(z^3 + 1\right)^{-1} dz$$

Part (A): if $r < \frac{\sqrt{3}}{2}$
Part (B): if $\frac{\sqrt{3}}{2} < r < 1$.

Problem 55. Let $R(x, y)$ be a rational function of x and y. The integral

$$I = \frac{1}{2\pi} \int_{-\pi}^{\pi} R(\cos\theta, \sin\theta)\, d\theta$$

may be put in the form

$$I = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(z) \frac{dz}{z}$$

where γ is the unit circle and $f(z)$ is a rational function of z. The trick is just to realize that on the unit cirle if z has phase θ then $2\cos\theta = z + 1/z$ and $2\mathrm{i}\sin\theta = z - 1/z$. Use this idea to calculate

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{2 + \cos\theta}.$$

Problem 56. Let Ω be an open subset of $\mathbb{C}$, $a \in \Omega$, and let $f : \Omega \to \mathbb{C}$ be analytic. Let $n \geq 0$ be an integer. Show there exists a unique function g_n analytic on Ω such that

$$f(z) - \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(a)(z - a)^k = (z - a)^n g_n(z)$$

for each $z \in \Omega$ (when $n = 0$ the sum is empty). Moreover show that

$$g_n(z) = \frac{1}{n!} f^{(n)}(a) + (z - a) g_{n+1}(z)$$

for each $z \in \Omega$ and each $n \geq 0$. Conclude

$$g_n(a) = \frac{1}{n!} f^{(n)}(a)$$

for each $n \geq 0$. Now let γ be a closed rectifiable curve in Ω and suppose that γ is null–homotopic in Ω (null–homologous if you want to use the stronger version of the CAUCHY theorem). Show that

$$\nu(\gamma, z) g_n(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(w)}{(w-a)^n (w-z)}\, dw$$

for each $z \in \Omega \sim \operatorname{traj} \gamma$. Thus show if $0 < r < \operatorname{dist}(a, \partial\Omega)$ and $M(r) = \max \{ \,|\, f(w) \,|\; \mid \; |\, w - a \,| = r \,\}$ then

$$|\, g_n(z) \,| \leq \frac{M(r)}{r^{n-1} \left(r - |\, z - a \,| \right)}$$

for $|\, z - a \,| < r$.

Problem 57. If f is an entire function with no zeros show that

$$|\, f(0) \,| > \min_{|\, w \,| = r} |\, f(w) \,|$$

for each $r > 0$. Use this result to deduce the Fundamental Theorem of Algebra.

Problem 58. Let $f : D(\alpha, R) \to \mathbb{C}$ be continuous. For each r with $0 < r < R$ let $\gamma_r : [a, b] \to D(\alpha, R)$ be defined by

$$\gamma_r(t) = \alpha + r\mathrm{e}^{\mathrm{i}t}.$$

Show that

$$\lim_{r \to 0} \frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} \frac{f(w)}{w - \alpha}\, dw = \frac{b-a}{2\pi} f(\alpha).$$

If f is analytic and $f^{(k)}(\alpha) = 0$ for $k = 0, 1, \cdots, n-1$ show that

$$\lim_{r \to 0} \frac{n!}{2\pi \mathrm{i}} \int_{\gamma_r} \frac{f(w)}{(w-\alpha)^{n+1}}\, dw = \frac{b-a}{2\pi} f^{(n)}(\alpha).$$

Note that γ_r need not be closed.

Remark 4.1. In the problems below you may assume (if necessary) that we know

$$\int_0^\infty \mathrm{e}^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

You should also be able to prove this fact. By a *contour* we mean a rectifiable curve. It is *simple* if it does not intersect itself (except at the initial point in the case of a closed contour).

Problem 59. Evaluate the integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \cos(2\,\omega x)\,dx$$

where $\omega > 0$ is a constant by integrating e^{-z^2} over the boundary of the rectangle with vertices $-\rho$, ρ, $\rho + \mathrm{i}\omega$ and $-\rho + \mathrm{i}\omega$ and letting $\rho \to \infty$. (Answer: $\sqrt{\pi}\,\mathrm{e}^{-\omega^2}$).

Problem 60. Evaluate the FRESNEL integrals

$$\int_0^{\infty} \cos\left(x^2\right)\,dx \qquad \int_0^{\infty} \sin\left(x^2\right)\,dx$$

by integrating e^{-z^2} over the boundary of the sector of the disk with center at the origin and radius ρ and aperature $\pi/4$ and then letting $\rho \to \infty$. (Answer: $\sqrt{\pi/8}$).

Problem 61. Evaluate the absolutely convergent integral

$$\int_0^{\pi} \log\left(\sin(x)\right)\,dx$$

by integrating $\log\left(1 - \mathrm{e}^{2\,\mathrm{i}z}\right)$ along the boundary of the rectangle with vertices $\mathrm{i}\rho$, 0, π and $\pi + \mathrm{i}\rho$ modified by excising the two lower corners and replacing that part of the contour by circular arcs of radius $\epsilon > 0$ with centers at the vertices. Then let $\rho \to \infty$ and $\epsilon \to 0$. Note that $(1 - \mathrm{e}^{2\,\mathrm{i}z}) \le 0$ only if $z = x + \mathrm{i}y$, $x = k\pi$ and $y \le 0$. Thus we may use the principal branch of the logarithm. In this case

$$\log\left(1 - \mathrm{e}^{2\,\mathrm{i}x}\right) = \log(2) + \log\left(\sin(x)\right) + \mathrm{i}\left(x - \frac{\pi}{2}\right), \qquad 0 < x < \pi.$$

(Answer: $-\pi \log 2$)

Problem 62. Show that

$$\lim_{\rho \to \infty} \int_0^{\rho} \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$$

by using the CAUCHY integral formula to evaluate

$$\int_{\gamma} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,dz$$

where γ is the simple closed contour consisting of semicircles with centers at the origin, in the upper half plane with radius ρ and in the lower half plane with radius ϵ, and portions of the diameter along the real axis and then letting $\rho \to \infty$ (you'll need the dominated convergence theorem) and letting $\epsilon \to 0$.

Problem 63. If $\beta > 0$ show that

$$\mathrm{e}^{-\beta} = \frac{2}{\pi} \int_0^{\infty} \frac{\cos\left(\beta x\right)}{1 + x^2}\,dx.$$

Do it by using the CAUCHY formula to evaluate the integral

$$\int_{\gamma} \frac{\mathrm{e}^{\mathrm{i}\beta z}}{1 + z^2}\,dz = \int_{\gamma} \frac{\mathrm{e}^{\mathrm{i}\beta z}/(z + \mathrm{i})}{z - \mathrm{i}}\,dz$$

where γ is the simple closed contour consisting of a semicircle of radius $\rho > 1$ in the upper half plane, with center at the origin, and of the diameter along the real axis. Then let $\rho \to \infty$.

5 Take-Home Exam

This test is a take–home examination. Please give clear and precise solutions to each problem and write–up your solutions neatly. You may discuss the problems with other people in general terms but you are expected to write–up your solutions on your own.

Problem 64. Let $f : D(0,1) \to \mathbb{C}$ be analytic and suppose $|\, f(z)\,| \le |\, z\,|^m$ for each z, $|\, z\,| < 1$, where $m \ge 1$ is an integer. Show that

either $|\, f(z)\,| < |\, z\,|^m$ for each $|\, z\,| < 1$, $z \ne 0$, and $\left|\, f^{(m)}(0)\,\right| < m!$

or there exists a constant c with $|\, c\,| = 1$ such that $f(z) = cz^m$ for each z.

Problem 65. Let f be continuous in $\overline{D(0,1)}$ and analytic in $D(0,1)$. Suppose $|\, f(z)\,| \ge \pi/3$ if $|\, z\,| = 1$ and suppose $f(0) = \frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$. Prove that f has a root in $D(0,1)$.

Problem 66. *From problem set 1 ⋯* If T is a Möbius transform with fixed points $a \ne b$ (both in $\mathbb{C}$) then there is a complex number $\lambda \ne 0$ such that

$$\frac{T^n(z) - a}{T^n(z) - b} = \lambda^n \, \frac{z - a}{z - b}$$

for each $z \in \mathbb{C}_\infty$ and each integer $n \ge 1$. **Hint:** Consider the cross–ratio $(T(z), \infty, a, b)$ to get the case $n = 1$. Then choose the Möbius transform R so $RTR^{-1}(w) = \lambda w$.

Problem 67. *From problem set 2 ⋯* Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where the power series has radius of convergence $R > 0$. Let $0 \le r < R$. Show that

$$\int_0^{2\pi} \left|\, f\left(r\mathrm{e}^{\mathrm{i}\theta}\right)\,\right|^2 \, d\theta = 2\pi \sum_{n=0}^{\infty} |\, a_n\,|^2 \, r^{2n}.$$

Conclude if $M(r) = \sup\limits_{|\, z\,| = r} |\, f(z)\,|$ then

$$|\, a_n\,| \le \frac{M(r)}{r^n}.$$

These inequalities are know as the Cauchy estimates. Write your proof so as to avoid using double sums.

Problem 68. *From problem set 3 …* Let f be an entire function. Suppose there exist a complex number α and real numbers K and m such that

$$\mathfrak{Re}\,(\alpha f(z)) \le K + m \log(1 + |\, z\,|)$$

for each $z \in \mathbb{C}$. Show that f is constant. **Hint:** The Fundamental Theorem of Algebra plays a role.

Problem 69. *From problem set 4* $\cdots$ Let γ be a circle of radius r parametrized in the usual way and suppose that γ does not pass through any of the roots of $z^3 + 1$. Find all possible values of

$$\int_\gamma \left(z^3 + 1\right)^{-1} dz$$

Part (A): if $r < \frac{\sqrt{3}}{2}$
Part (B): if $\frac{\sqrt{3}}{2} < r < 1$.

Please put each solution on a separate sheet of paper. Label each sheet with your name and the problem number. If you turn in a solution with steps missing, indicate that you are aware of the missing steps, or sketch how you thought you might fill them in. Be precise and careful, even excessively so. This is a test – you are not being asked to produce stylish mathematical writing!
Try to enjoy the problems! Problems 2 (65) and 5 (68) have easy short solutions (once you figure out what to do – which is the hard part).

6 Contact Information

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