Complex Analysis I – Mth 514

Archive – Fall 1993 Files

Feb 18, 2001

This archive contains the assignments and some answers from Mth 514 Fall 1993.

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1 Assignment A

Problem 1. For each integer $n > 1$ let ξ_n be any primitive n^{th} root of unity. Compute

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \xi_n^k - 1 \right| .$$

Problem 2. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series which converges for all z and suppose that $a_n \neq 0$ for each n. Let

$$p_n(z) = \begin{cases} a_0 & \text{if } n = 0 \\ a_n z^n - a_{n-1} z^{n-1} & \text{if } n \geq 1 \end{cases}$$

(A) Show that the polynomials $p_n(z)$ are linearly independent. **(B)** Find complex numbers b_n, not all zero, such that

$$\sum_{n=0}^{\infty} b_n \, p_n(z) = 0 \text{ for each } z.$$

2 Answers A

Problem 1 solution. For any real θ

$$\mathrm{e}^{\mathrm{i}\theta} - 1 = \mathrm{e}^{\mathrm{i}\theta/2}\left(\mathrm{e}^{\mathrm{i}\theta/2} - \mathrm{e}^{-\mathrm{i}\theta/2}\right) = 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta/2}\,\sin\left(\frac{\theta}{2}\right).$$

Thus for $0 \leq \theta \leq 2\pi$ we have

$$\left|\,\mathrm{e}^{\mathrm{i}\theta} - 1\,\right| = 2\sin\left(\frac{\theta}{2}\right).$$

Without loss of generality $\xi_n = \mathrm{e}^{2\pi\mathrm{i}/n}$ and so $\left|\,\xi_n^k - 1\,\right| = 2\sin\left(\frac{k\pi}{n}\right)$. Thus

$$\sum_{k=1}^{n}\left|\,\xi_n^k - 1\,\right| = \sum_{k=1}^{n} 2\sin\left(\frac{k\pi}{n}\right) = \sum_{k=0}^{n-1} 2\sin\left(\frac{k\pi}{n}\right) = 2\,\Im\mathfrak{m}\left(\sum_{k=0}^{n-1}\mathrm{e}^{k\pi\mathrm{i}/n}\right)$$

Since the last sum is a geometric sum we have

$$= 2\,\Im\mathfrak{m}\left(\frac{1-\mathrm{e}^{\pi\mathrm{i}}}{1-\mathrm{e}^{\pi\mathrm{i}/n}}\right) = \frac{2}{\sin\left(\frac{\pi}{2n}\right)}\,\Im\mathfrak{m}\left(\frac{2}{-2\mathrm{i}\mathrm{e}^{\frac{\pi\mathrm{i}}{2n}}}\right) = \frac{-2}{\sin\left(\frac{\pi}{2n}\right)}\,\Re\mathfrak{e}\left(\mathrm{e}^{-\frac{\pi\mathrm{i}}{2n}}\right) = \frac{2}{\tan\left(\frac{\pi}{2n}\right)}.$$

Since $\frac{x}{\tan(x)} \to 1$ as $x \to 0$ we see that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\,\xi_n^k - 1\,\right| = \frac{4}{\pi}.$$

Comment: The purpose of this problem was to provide practice in doing complex arithmetic. Some people made use of the fact that we are really computing the limit of a sequence of Riemann sums (thereby bypassing all of the arithmetic).

Problem 2 solution. Since $a_n \neq 0$ for each n we see that $p_n(z)$ has degree exactly n. Thus linear indepence is obvious. For the last part choose $b_n = 1$ for each n. Then

$$\sum_{k=0}^{n} b_k \, p_k(z) = a_n \, z^n \to 0$$

since

$$\sum_{n=0}^{\infty} a_n z^n$$

is convergent. Thus

$$\sum_{k=0}^{\infty} b_k\, p_k(z) = 0$$

as required.

Comment: Note that $\sum_{n=0}^{\infty}\big((n+1)-n\big) = \infty$ but every term 'cancels' $1+(2-1)+(3-2)+(4-3)+\cdots = 1-1+2-2+3-3+4-\ldots$. Cancellation arguments need some justification. Manipulating power series is also dangerous. For example,

$$\sum_{n=0}^{\infty}\left(n\,!+\frac{1}{n\,!}\right) z^n - \sum_{n=0}^{\infty} n\,!\, z^n = \mathrm{e}^z$$

is *not* a valid conclusion. We can *carelessly* manipulate power series only after checking convergence.

3 Assignment B

Problem 3. Let $a_n \neq 0$ for each n. Show that

$$\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \le \liminf_{n\to\infty}|a_n|^{1/n} \le \limsup_{n\to\infty}|a_n|^{1/n} \le \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

Problem 4. Let $\sum_{n=0}^{\infty} b_n$ be any series such that $b_n \neq 0$ for each n and such that $\rho = \lim_{n\to\infty}|b_n|^{1/n}$ exists. Define

$$a_n = 2^{(-1)^n}\, b_n.$$

Show that the root test for the series $\sum a_n$ works if $\rho \neq 1$ but the ratio test fails if $\frac{1}{4} \le \rho \le 4$. By using, for example, geometric series for $\sum b_n$, give an example of a convergent and a divergent series where the root test works but the ratio test fails.

4 Answers B

Problem 3 solution. Let

$$A = \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

If $A < t$ then $\left|\frac{a_{n+1}}{a_n}\right| < t$ for all but finitely many n. Hence there is N such that $|\,a_n\,| \le t^{n-N}\,|\,a_N\,|$ for all $n \ge N$. It follows that $\limsup_{n\to\infty}|\,a_n\,|^{1/n} \le t$. We have proved this last inequality for any $t > A$. Thus $\limsup_{n\to\infty}|\,a_n\,|^{1/n} \le A$.

The middle inequality is trivial. The remaining inequality is proved in a manner similar to the above.

Comment: The hypothesis $A < t$ is vacuous when $A = \infty$ so it's not really necessary to give a separate argument in this case. In considering the inequality for the lim inf's a number of people fussed about $-\infty$. Again it's not really necessary in general. Moreover it's downright strange, when dealing with a sequence of nonnegative numbers.

Problem 4 solution. By problem 3 and by construction

$$\rho_1 = \limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \geq 4 \limsup_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| \geq 4 \limsup_{n\to\infty} |\, b_n \,|^{1/n} = 4\rho$$

and

$$\rho_0 = \liminf_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \leq \frac{1}{4} \liminf_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| \leq \frac{1}{4} \liminf_{n\to\infty} |\, b_n \,|^{1/n} = \frac{1}{4}\rho.$$

Moreover

$$\limsup_{n\to\infty} |\, a_n \,|^{1/n} = \limsup_{n\to\infty} |\, b_n \,|^{1/n}$$

since

$$\lim_{n\to\infty} 2^{\frac{(-1)^n}{n}} = 1.$$

Thus if $\rho \neq 1$ the root test works for $\sum_{n=0}^{\infty} a_n$. If $1/4 \leq \rho \leq 4$ then $\rho_1 \geq 1$ and $\rho_0 \leq 1$. Thus the ratio test fails. Taking $b_n = 2^{-n}$ yields a convergent series $\sum a_n$ for which the ratio test fails ($\rho = 1/2$) and taking $b_n = 2^n$ yields a divergent series $\sum a_n$ for which the ratio test fails ($\rho = 2$).

Comment: A number of people tried to argue with limits. When the limits are not known to exist such arguments are not valid and may lead to erroneous conclusions. Some people were not clear on the statement of the ratio test: if $\sum a_n$ is a series in which $a_n \neq 0$ for each n and

$$\rho_0 = \liminf \left| \frac{a_{n+1}}{a_n} \right| \quad \text{and} \quad \rho_1 = \limsup \left| \frac{a_{n+1}}{a_n} \right|$$

then

- $\rho_0 > 1$ implies the series diverges
- $\rho_1 < 1$ implies the series converges absolutely
- $\rho_0 \leq 1$ *and* $\rho_1 \geq 1$ implies the test fails.

Note in the last conclusion it is the conjuction of the two conditions which is assumed. Either condition by itself does not imply failure of the test. For example $\rho_0 \leq \rho_1 < 1$ implies absolute convergence (but $\rho_0 \leq 1$).

5 Assignment C

Problem 5. Show

$$\left|e^{i\theta} - e^{i\phi}\right| = 2\sin\left|\frac{\theta-\phi}{2}\right|, \quad \text{if } |\theta-\phi| \le \pi.$$

Use this result to prove

$$\frac{1}{2}\,|\theta-\phi| \le \left|e^{i\theta} - e^{i\phi}\right| \le |\theta-\phi|, \quad \text{if } |\theta-\phi| \le \frac{\pi}{2}.$$

Now prove that

$$g(z) = \log(|z|) + i\arg(z), \quad -\pi < \arg(z) < \pi,$$

is continuous in

$$\mathbb{C} - \{\, z \in \mathbb{R} \;\mid\; z \le 0 \,\}.$$

Problem 6. The proof of the differentiability of the sum of a convergent power series may be based on the identity

$$\frac{z^k - w^k}{z-w} - kw^{k-1} = \begin{cases} 0 & \text{if } k = 1 \\ (z-w)\sum_{j=1}^{k-1} jw^{j-1}z^{k-j-1} & \text{if } k \ge 2. \end{cases}$$

Prove this identity. Then use it to estimate

$$\left|\frac{f(z)-f(w)}{z-w} - g(w)\right|$$

where $|z| < r$, $|w| < r$, $0 < r < R$, $f(z) = \sum a_n z^n$ has radius of convergence R and $g(z) = \sum n a_n z^{n-1}$. Use your estimate to prove the differentiability theorem.

6 Assignment D

Problem 7. Use the theorem on differentiation of power series to sum the series

$$\sum_{n=1}^{\infty} n^2 z^n.$$

Problem 8. Show Newton's binomial series

$$\sum_{n=0}^{\infty} \binom{w}{n} z^n, \qquad \binom{w}{n} = \frac{w(w-1)\cdots(w-n+1)}{n!}$$

has radius of convergence 1.

Problem 9. Let Ω be an open subset of $\mathbb{C}$. Let $\gamma(t) = \alpha(t) + i\beta(t)$, $t \in [a,b]$, be a C^1 curve in Ω with tangent vector $\gamma'(t) = \alpha'(t) + i\beta'(t)$. Let $f : \Omega \to \mathbb{C}$ be Frechét differentiable on Ω. Show that

$$\frac{d}{dt}(f\circ\gamma) = \left(\frac{\partial f}{\partial z}\circ\gamma\right)\gamma' + \left(\frac{\partial f}{\partial \bar{z}}\circ\gamma\right)\overline{\gamma'}$$

where $\frac{\partial}{\partial \bar{z}}$ is the Cauchy–Riemann operator and $\frac{\partial}{\partial z}$ is its conjugate. In particular if f is (complex) differentiable in Ω then

$$\frac{d}{dt}(f\circ\gamma) = (f'\circ\gamma)\;\gamma'.$$

7 Assignment E

Let $\mathbb{M}$ be the group of Möbius transformations of $\hat{\mathbb{C}}$ and let $\Theta : \mathsf{GL}(2, \mathbb{C}) \to \mathbb{M}$ be the canonical homomorphism of groups.

Problem 10. If A, $B \in \mathsf{GL}(2, \mathbb{C})$ then $\Theta(A) = \Theta(B)$ if and only if there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that $A = \lambda B$. Thus if $S \in \mathbb{M}$ there are 2 matrices $A \in \mathsf{GL}(2, \mathbb{C})$ such that $S = \Theta(A)$ and $\det(A) = 1$.

Problem 11. If $S \in \mathbb{M}$ then $S(\hat{\mathbb{R}}) = \hat{\mathbb{R}}$ if and only if there exists $A \in \mathsf{GL}(2, \mathbb{R})$ such that $\Theta(A) = S$. Moreover, in this case there are 2 matrices $A \in \mathsf{GL}(2, \mathbb{R})$ such that $S = \Theta(A)$ and $\det(A) = 1$ or $\det(A) = -1$.

Hint: $(S(z), S(a_1), S(a_2), S(a_3)) = (z, a_1, a_2, a_3)$.

Problem 12. If $a \neq b$ are complex numbers and $\mu > 0$ then show

$$\left| \frac{z-a}{z-b} \right| = \mu$$

is a circle Γ_μ in $\hat{\mathbb{C}}$. (It is a line if $\mu = 1$). The circles Γ_μ are called the circles of Apollonius with limit points a and b. Show that a and b are Γ_μ–symmetric. Conversely show if Γ is any circle such that a and b are Γ–symmetric then $\Gamma = \Gamma_\mu$ for some $\mu > 0$.

Hint: Consider $T \in \mathbb{M}$ defined by $T(z) = \frac{z-a}{z-b}$.

Problem 13. If $S \in \mathbb{M}$ and $a \neq b$ are complex numbers show if S has fixed points a and b then

$$\frac{S(z)-a}{S(z)-b} = \lambda \, \frac{z-a}{z-b}$$

for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Now show if $n \in \mathbb{Z}$ then

$$\frac{S^n(z)-a}{S^n(z)-b} = \lambda^n \, \frac{z-a}{z-b}.$$

Hint: Compute $(S(z), \infty, a, b)$.

8 Assignment F

Let $\mathbb{M}$ be the group of Möbius transformations of $\hat{\mathbb{C}}$ and let $\Theta : \mathsf{GL}(2, \mathbb{C}) \to \mathbb{M}$ be the canonical homomorphism of groups.

Problem 14. Find $T \in \mathbb{M}$ such that T maps the right half plane $\{\, z \in \mathbb{C} \mid \Re z > 0 \,\}$ onto the unit disk $\{\, z \in \mathbb{C} \mid |z| < 1 \,\}$ and $T(1) = 0$. How do we know that $T(-1) = \infty$ without computing T?

Problem 15. If $S, T \in \mathbb{M}$ we say S and T are *equivalent* if there exists $R \in \mathbb{M}$ such that

$$S = R^{-1}TR.$$

If $T \in \mathbb{M}$,

$$T(z) = \frac{az+b}{cz+d}, \quad \text{that is, } \Theta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T,$$

then define $\kappa(T)$ by

$$\kappa(T) = \frac{(a+d)^2}{ad-bc} - 2 = \frac{\Delta}{ad-bc} + 2$$

where

$$\Delta = (a-d)^2 + 4bc$$

is the discriminant of the quadratic equation for the fixed points

$$cz^2 + (a-d)z - b = 0.$$

Note that κ is well-defined, that is, does not depend on the choice of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

to represent T. **(A)** Show that $T \neq I$ has one fixed point of multiplicity 2 if and only if $\kappa(T) = 2$, if and only if T is equivalent to the translation operator S defined by $S(z) = z + 1$. **(B)** If $a \in \mathbb{C}$, $a \neq 1$, then T is equivalent to the homothety S defined by $S(z) = az$ if and only if $\kappa(T) = a + a^{-1}$. **(C)** Finally show if S and T are non-identity MÖBIUS transfomations then S and T are equivalent if and only if $\kappa(T) = \kappa(S)$.

Hint: Recall some linear algebra, especially the JORDAN canonical form for similarity equivalence.

Problem 16. Let $c > \rho > 0$, let Γ be the circle with center at c and radius ρ and let Γ' be the circle with center at $-c$ and radius ρ. Find $T \in \mathbb{M}$ such that $T(\Gamma)$ and $T(\Gamma')$ are concentric circles with centers at the origin.

Hint: If $a = \sqrt{(c^2 - \rho^2)}$ then a and $-a$ are Γ–symmetric and Γ'–symmetric. Now choose T with $T(-a) = \infty$, $T(a) = 0$ and $T(0) = -1$.

Problem 17. Let $c > \rho > 0$, let Γ be the circle with center at c and radius ρ and let L be the imaginary axis. Find $T \in \mathbb{M}$ such that $T(\Gamma)$ and $T(L)$ are concentric circles with centers at the origin.

Hint: If $a = \sqrt{(c^2 - \rho^2)}$ then a and $-a$ are Γ–symmetric and L–symmetric.

Problem 18. Prove DIRAC's formula: if ϕ is a continuously differentiable function on $\mathbb{R}$ and ϕ vanishes outside of a compact set then

$$\lim_{y\downarrow 0} \int_{-\infty}^{\infty} \frac{\phi(x)}{x+iy}\,dx = -i\pi\phi(0) + \lim_{\epsilon\downarrow 0} \int_{|\,x\,|>\epsilon} \frac{\phi(x)}{x}\,dx.$$

Hint: Define $\log(z) = \log(|\,z\,|) + i\arg(z)$ where $-\pi/2 < \arg(z) < 3\pi/2$ and note if $y > 0$ then

$$\int_{-\infty}^{\infty} \frac{\phi(x)}{x+iy}\,dx = -\int_{-\infty}^{\infty} \log(x+iy)\phi'(x)\,dx.$$

9 Assignment G

The three problems in this assignment are to be done without using Cauchy's theorem. Just use the definition of the contour integral and the techniques and results that we discussed in the first few examples.

Problem 19. Let γ be a circle of radius r parametrized in the usual way and suppose that γ does not pass through any of the roots of $z^3 + 1$. Find all possible values of

$$\int_\gamma \left(z^3 + 1\right)^{-1} dz$$

if **(A)** $r < \frac{\sqrt{3}}{2}$ and **(B)** $\frac{\sqrt{3}}{2} < r < 1$.

Problem 20. Let $R(x, y)$ be a rational function of x and y. The integral

$$I = \frac{1}{2\pi} \int_{-\pi}^{\pi} R(\cos\theta, \sin\theta)\, d\theta$$

may be put in the form

$$I = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(z)\, \frac{dz}{z}$$

where γ is the unit circle and $f(z)$ is a rational function of z. The trick is just to realize that on the unit cirle if z has phase θ then $2\cos\theta = z + 1/z$ and $2\mathrm{i}\sin\theta = z - 1/z$. Use this idea to calculate

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{2 + \cos\theta}.$$

Problem 21. Let γ be the unit circle parametrized in the usual way. Compute the integrals

(A) $\displaystyle\int_\gamma \overline{z}\, dz$ **(B)** $\displaystyle\int_\gamma \overline{z}\, |\, dz\, |.$

10 Assignment H

In the problems below you may assume we know

$$\int_0^\infty \mathrm{e}^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

You should in fact also be able to prove this fact.

Problem 22. Evaluate the integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \cos(2\,\omega x)\, dx$$

where $\omega > 0$ is a constant by integrating e^{-z^2} over the boundary of the rectangle with vertices $-\rho$, ρ, $\rho + i\omega$ and $-\rho + i\omega$ and letting $\rho \to \infty$. (Answer: $\sqrt{\pi}\, e^{-\omega^2}$).

Problem 23. Evaluate the Fresnel integrals

$$\int_0^\infty \cos\left(x^2\right)\, dx \qquad \int_0^\infty \sin\left(x^2\right)\, dx$$

by integrating e^{-z^2} over the boundary of the sector of the disk with center at the origin and radius ρ and aperature $\pi/4$ and then letting $\rho \to \infty$. (Answer: $\sqrt{\pi/8}$).

Problem 24. Evaluate the absolutely convergent integral

$$\int_0^\pi \log\left(\sin(x)\right)\, dx$$

by integrating $\log\left(1 - e^{2\,iz}\right)$ along the boundary of the rectangle with vertices $i\rho$, 0, π and $\pi + i\rho$ modified by excising the two lower corners and replacing that part of the contour by circular arcs of radius $\epsilon > 0$ with centers at the vertices. Then let $\rho \to \infty$ and $\epsilon \to 0$. Note that $(1 - e^{2\,iz}) \leq 0$ only if $z = x + iy$, $x = k\pi$ and $y \leq 0$. Thus we may use the principal branch of the logarithm. In this case

$$\log\left(1 - e^{2\,ix}\right) = \log(2) + \log\left(\sin(x)\right) + i\left(x - \frac{\pi}{2}\right), \qquad 0 < x < \pi.$$

(Answer: $-\pi \log 2$)

Problem 25. Show that

$$\lim_{\rho\to\infty} \int_0^\rho \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}$$

by using Cauchy's integral formula to evaluate

$$\int_\gamma \frac{e^{iz}}{z}\, dz$$

where γ is the simple closed contour consisting of semicircles with centers at the origin, in the upper half plane with radius ρ and in the lower half plane with radius ϵ, and portions of the diameter along the real axis and then letting $\rho \to \infty$ (you'll need the dominated convergence theorem) and letting $\epsilon \to 0$.

Problem 26. If $\beta > 0$ show that

$$e^{-\beta} = \frac{2}{\pi} \int_0^\infty \frac{\cos\left(\beta x\right)}{1 + x^2}\, dx.$$

Do it by using the Cauchy formula to evaluate the integral

$$\int_\gamma \frac{e^{i\beta z}}{1 + z^2}\, dz = \int_\gamma \frac{e^{i\beta z}/(z + i)}{z - i}\, dz$$

where γ is the simple closed contour consisting of a semicircle of radius $\rho > 1$ in the upper half plane, with center at the origin, and of the diameter along the real axis. Then let $\rho \to \infty$.

11 Contact Information

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