Advanced Engineering Mathematics – Mth 481/581

Archive – Winter 1992 Files

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This archive contains the tests from Mth 481 Winter 1992. The original test instructions, headers and formatting have not been preserved.

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1 Test 1

Problem 1. Find the radius of convergence of the power series

$$\text{(A)} \qquad \sum_{n=0}^{\infty} \frac{3^n n}{8^{n/2}}\, x^n \qquad\qquad \text{(B)} \qquad \sum_{n=0}^{\infty} \frac{2^{2n+3} n^{3n}}{(3n)!}\, x^n.$$

Problem 2. Given $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x and

$$f(x) = -x + \frac{x^2}{4} - \frac{x^3}{7} + \dots$$

for $|x| < 1$ find the first four terms in the Taylor-Maclaurin series of $e^{f(x)}$.

Problem 3. Find the first 10 terms in the power series solution to the initial value problem

$$y'' + x^2 y = 0, \qquad y(0) = k_1, \quad y'(0) = k_2$$

(i.e., all terms up through degree 9).

Problem 4. The ordinary differential equation

$$x^2y'' + xy' - (1 + x^2)y = 0$$

has a *regular singular point* at the origin. Find the indicial equation and the indicial roots. Then use the method of Frobenius to find a solution to the equation as a *fractional power series*. It is sufficient to give the first four nonzero terms in the series.

Problem 5. Solve the initial value problem

$$y''' + 2y'' - y' - 2y = 0, \qquad y(0) = 4, \quad y'(0) = 3, \quad y''(0) = 7.$$

Problem 6. Find a *particular* solution to the linear ordinary differential equation

$$y''' - 3y'' + 3y' - y = (x^2 + 1)\, e^x.$$

2 Test 2

The first three problems refer to the following regular STURM-LIOUVILLE problem

$$(*) \quad \begin{cases} y'' + \lambda y = 0 \\ y(0) + y'(0) = 0 \\ y(1) + y'(1) = 0 \end{cases}$$

Problem 7. The STURM-LIOUVILLE problem $(*)$ has one negative eigenvalue. Find this negative eigenvalue and the corresponding eigensolution.

Problem 8. The positive eigenvalues of $(*)$ are given by $\lambda_k = k^2\pi^2$, where $k \geq 1$ is an integer. Find the corresponding eigensolutions.

Problem 9. Is 0 an eigenvalue of $(*)$?

Problem 10. Let $(f_n)_{n\geq 1}$ be a complete orthogonal sequence of continuous real functions on the interval $[a, b]$ with respect to the weight function p. Here p is a continuous and positive function on $[a, b]$. Let

$$||f_n|| = \left(\int_a^b |f_n(x)|^2 p(x) dx \right)^{1/2}.$$

Let g be a continuous function on $[a, b]$ and let $b_n = \int_a^b g(x) f_n(x) p(x) dx$. Find the coefficients a_n in the orthogonal expansion of g

$$g(x) = \sum_{n=1}^{\infty} a_n f_n(x).$$

Problem 11. The differential equation

$$x^2y'' - 2xy' + 2y = 4x^2 + 2x^3$$

has complementary solution $c_1x + c_2x^2$. Use variation of parameters to find the general solution.

3 Final Exam

Problem 12. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n^3}{5^n}\, x^{2n}.$$

Problem 13. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n^n}{n!\, 3^n}\, x^n.$$

Problem 14. Let $y = \sum_{n=0}^{\infty} a_n x^n$ be a power series solution of the ordinary differential equation

$$(1 + 3x^2)y'' - x^2 y' + xy = 0.$$

How large is the radius of convergence R of this power series guaranteed to be? (I.e. find a lower bound for R. Be sure to justify your answer.)

Problem 15. Find the first 10 terms in the power series solution to the initial value problem

$$y'' - x^3 y = 0, \qquad y(0) = k_1, \quad y'(0) = k_2$$

(i.e., all terms up through degree 9).

Problem 16. If we apply the method of FROBENIUS to find a fractional power series solution $\sum_{n=0}^{\infty} a_n x^{n+\lambda}$, ($a_0 \neq 0$) to the ordinary differential equation

$$8x^2 y'' + 10x(1 - x)y' - 3y = 0$$

we end up with the indicial equation plus some recurrence relations for the coefficients a_n. Find the indicial equation and compute the indicial roots.

Problem 17. Suppose we have a homogeneous linear ordinary differential equation with constant real coefficients and we determine it has characteristic roots

$$2 + 3i, 2 + 3i, 2 - 3i, 2 - 3i, -2, -2, -2, 1, 0, 0.$$

Here we have repeated each root according to its multiplicity. What is the general solution (in real form) of our differential equation?

Problem 18. FInd the general solution of the linear ordinary differential equation

$$y'' + 2y' - 3y = (x^2 + 2)e^x.$$

Problem 19. FInd the general solution of the linear ordinary differential equation

$$y'' - 4y' + 13y = e^{2x}\cos(3x).$$

Problem 20. Find a particular solution of the linear ordinary differential equation

$$y'' + y = \sec(x)\tan(x).$$

You might (or might not) find the following integrals useful:

$$\int \tan x\,dx = \log|\cos x|, \quad \int \tan^2 x\,dx = -x + \tan x, \quad \int \sec x\,dx = \log|\sec x + \tan x|.$$

Problem 21. Find all the eigenvalues and the corresponding eigensolutions for the regular STURM-LIOUVILLE problem

$$(e^{4x}y')' + \lambda e^{4x}y = 0, \qquad y(0) = 0, \quad y(1) = 0.$$

Problem 22. Consider the plane autonomous system

$$\frac{dx}{dt} = (x^2 - 1)y, \qquad \frac{dy}{dt} = (y^2 - 1)x.$$

1. Find all of the equilibrium points.
2. At each equilibrium point (x_0, y_0) compute the JACOBI matrix $J(x_0, y_0)$.
3. Classify the equilibrium point at the origin.

By the JACOBI matrix we mean the coefficient matrix of the linearized system.

4 Extra Assignment for Mth 581 Students

Problem 23. Apply the GRAMM–SCHMIDT process to orthonormalize the polynomials $1, x, x^2, x^3, \ldots$ on the interval $[0, \infty)$ with respect to the weight function $\exp(-x)$. Compute the first three orthonormal polynomials explicitly.

Problem 24. Find all the eigenvalues and the corresponding eigensolutions for the regular STURM-LIOUVILLE problem

$$(xy')' + \left(\frac{\lambda}{x}\right)y = 0, \qquad y(1) = y(2) = 0.$$

If f is a piecewise continuously differentiable function on $[1, 2]$ we know

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$$

where the y_n are the eigensolutions. Give an explicit integral formula for the a_n.

Problem 25. Find all the eigenvalues and the corresponding eigensolutions for the regular STURM-LIOUVILLE problem

$$(e^x y')' + \lambda e^x y = 0, \qquad y(0) = y(T) = 0$$

where $T > 0$ is some given number.

Problem 26. Apply the method of FROBENIUS

$$y = \sum_{n=0}^{\infty} a_n x^{n+\lambda}, \qquad a_0 \neq 0$$

to the ordinary differential equation

$$8x^2 y'' + 10x(1-x)y' - 3y = 0.$$

Find the indicial roots and the first few terms in the series solutions.

Problem 27. For the plane autonomous system

$$\frac{dx}{dt} = (x-1)x(y+2), \qquad \frac{dy}{dt} = (x+2)y$$

find all of the equilibrium points. Classify each equilibrium point (as far as possible) by investigating the linearization at each equilibrium point.

5 Contact Information

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