Advanced Engineering Mathematics – Mth 481/581

Archive – Summer 1994 Files

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This archive contains the sample problems and tests from Mth 481/581 Summer 1994. The original test instructions, headers and formatting have not been preserved.

Contents

1 Sample Problems 1

Problem 1. Given $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x and

$$f(x) = -x + \frac{x^2}{4} - \frac{x^3}{7} + \dots$$

for $|x| < 1$ find the first four terms in the Taylor-Maclaurin series of $e^{f(x)}$.

Problem 2. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n^3}{5^n} x^{2n}.$$

Problem 3. Find the radius of convergence of the power series

$$\text{(A)} \quad \sum_{n=0}^{\infty} \frac{3^n n}{8^{n/2}}\, x^n \qquad\qquad \text{(B)} \quad \sum_{n=0}^{\infty} \frac{2^{2n+3} n^{3n}}{(3n)!}\, x^n.$$

Problem 4. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n^n}{n!\, 3^n}\, x^n.$$

Problem 5. Let $y = \sum_{n=0}^{\infty} a_n x^n$ be a power series solution of the ordinary differential equation

$$(1 + 3x^2)y'' - x^2 y' + xy = 0.$$

How large is the radius of convergence R of this power series guaranteed to be? (I.e. find a lower bound for R. Be sure to justify your answer.)

Problem 6. Find the first 10 terms in the power series solution to the initial value problem

$$y'' + x^2 y = 0, \qquad y(0) = k_1, \quad y'(0) = k_2$$

(i.e., all terms up through degree 9).

Problem 7. Find the first 10 terms in the power series solution to the initial value problem

$$y'' - x^3 y = 0, \qquad y(0) = k_1, \quad y'(0) = k_2$$

(i.e., all terms up through degree 9).

Problem 8. If we apply the method of FROBENIUS to find a fractional power series solution $\sum_{n=0}^{\infty} a_n x^{n+\lambda}$, ($a_0 \neq 0$) to the ordinary differential equation

$$8x^2 y'' + 10x(1 - x)y' - 3y = 0$$

we end up with the indicial equation plus some recurrence relations for the coefficients a_n. Find the indicial equation and compute the indicial roots.

Problem 9. The ordinary differential equation

$$x^2 y'' + xy' - (1 + x^2)y = 0$$

has a *regular singular point* at the origin. Find the indicial equation and the indicial roots. Then use the method of FROBENIUS to find a solution to the equation as a *fractional power series*. It is sufficient to give the first four nonzero terms in the series.

Problem 10. Suppose we have a homogeneous linear ordinary differential equation with constant real coefficients and we determine it has characteristic roots

$$2 + 3i, 2 + 3i, 2 - 3i, 2 - 3i, -2, -2, -2, 1, 0, 0.$$

Here we have repeated each root according to its multiplicity. What is the general solution (in real form) of our differential equation?

Problem 11. FInd the general solution of the linear ordinary differential equation

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = (x^2+2)\mathrm{e}^{-x}.$$

Problem 12. FInd the general solution of the linear ordinary differential equation

$$y'' - 4y' + 13y = \mathrm{e}^{2x}\cos(3x).$$

Problem 13. Find a particular solution of the linear ordinary differential equation

$$y'' + y = \sec(x)\tan(x).$$

You might (or might not) find the following integrals useful:

$$\int \tan x\,dx = -\log|\cos x|, \quad \int \tan^2 x\,dx = -x + \tan x, \quad \int \sec x\,dx = \log|\sec x + \tan x|.$$

Problem 14. Find the radius of convergence of the power series

$$\textbf{(A)} \quad \sum_{n=0}^{\infty} \frac{3^n n^{3n}}{(2n)!\,n!}\,x^n \qquad \textbf{(B)} \quad \sum_{n=0}^{\infty} \frac{3^n n^{3n}}{(2n)!\,n!}\,x^{2n}.$$

Problem 15. Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

be a power series solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} + x^3\frac{dy}{dx} + 3x^2 y = 0.$$

If we compute the coefficients a_n in the power series expansion of the solution $y(x)$ we find that

$$y(x) = a_0 \mathrm{e}^{-x^4/4} + a_1\,y_1(x).$$

Find the power series representation of $y_1(x)$. Give enough terms to show the pattern, or give the n^{th} term.

Problem 16. The ordinary differential equation

$$9x^2\frac{d^2y}{dx^2} + 9x^2\frac{dy}{dx} + 2y = 0$$

has a regular singular point at the origin. Find the indicial equation and the indicial roots. Then use the method of Frobenius to find linearly independent *fractional power series* solutions to the equation. *(Just a few terms, say four each.)*

Problem 17. For a certain fourth order linear ordinary differential equation with constant coefficients,

$$L[y] = x\mathrm{e}^{-x}\cos(x) + x^2\mathrm{e}^{-x} + x$$

the characteristic polynomial is

$$\left(\lambda^2 + 2\lambda + 5\right)^2.$$

(A) Find the complimentary solution. **(B)** What form should the particular solution take, according to the method of undetermined coefficients? *(Do not compute the undetermined coefficients. Life's too short.)*

Problem 18. Solve the ordinary differential equation

$$9x^2\frac{d^2y}{dx^2} + 2y = 0.$$

2 Sample Problems 2

Problem 19. Consider the plane autonomous system

$$\frac{dx}{dt} = (x^2-1)y, \qquad \frac{dy}{dt} = (y^2-1)x.$$

1. Find all of the equilibrium points.
2. At each equilibrium point (x_0, y_0) compute the Jacobi matrix $J(x_0, y_0)$.
3. Classify the equilibrium point at the origin.

By the Jacobi matrix we mean the coefficient matrix of the linearized system.

Problem 20. Compute e^{tA} where

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Problem 21. Given

$$\mathrm{e}^{tA} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

find a particular solution to the system of ordinary differential equations

$$\frac{d\vec{x}}{dt} = A\vec{x} + \begin{bmatrix} 1 \\ \cos(t) \end{bmatrix}.$$

Problem 22. Find the general solution (in real form)

$$\begin{aligned} \frac{dx_1}{dt} + \frac{dx_2}{dt} + 2x_1 - x_2 &= 0 \\ \frac{dx_1}{dt} - \frac{dx_2}{dt} + x_1 - 2x_2 &= 0 \end{aligned}$$

Problem 23. Compute the exponential e^{tA} given

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix}.$$

Problem 24. Given

$$e^{tA} = \frac{1}{7}\begin{bmatrix} 6e^{3t} + e^{-4t}, & e^{3t} - e^{-4t} \\ 6e^{3t} - 6e^{-4t}, & e^{3t} + 6e^{-4t} \end{bmatrix}$$

(A) solve the initial value problem

$$\frac{d\vec{x}}{dt} = A\vec{x} + \begin{bmatrix} 1 \\ e^{3t} \end{bmatrix}, \qquad \vec{x}(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

and **(B)** find the matrix A.

Problem 25. Consider the system

$$\begin{cases} \dfrac{dx}{dt} = x\left(1 - y^2\right) \\ \dfrac{dy}{dt} = (x+1)(y+2). \end{cases}$$

(A) Find all of the equilibrium points. **(B)** Classify the equilibrium points (to the extent possible) by studying the Jacobi matrix.

Problem 26. Suppose $A^3 = 0$. **(A)** Show e^{tA} is a polynomial in t. **(B)** Compute e^{tA} if

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

3 Test 1

Problem 27. Suppose we have a homogeneous linear ordinary differential equation with constant *real* coefficients and we determine its characteristic roots to be

$$1 + 2i,\ 1 + 2i,\ 1 - 2i,\ 1 - 2i,\ 3,\ 3\ 3,\ 0,\ 0$$

where we have repeated each root according to its multiplicity. What is the general solution (in *real form*) of our ordinary differential equation.

Problem 28. The homogeneous ordinary differential equation

$$x^3 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

has the general solution

$$y = c_1 x + c_2 x e^{-1/x}.$$

Use variation of parameters to find a particular solution of the inhomogeneous ordinary differential equation

$$x^3 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \mathrm{e}^{-1/x}.$$

Problem 29. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{5^n n^{4n}}{((2n)!)^2} x^{2n}$$

Problem 30. Find all terms of degree less than 20 in the power series (about 0) solution to the initial value problem

$$\frac{d^2y}{dx^2} + x^6 y = 0, \quad y(0) = k_1, \quad y'(0) = k_2.$$

Problem 31. The ordinary differential equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - \left(\frac{1}{4} + x\right) y = 0$$

has a regular singular point at the origin. Find the indicial equation and the indicial roots. Then use the method of Frobenius with *one* of the indicial roots to find a solution to the equation as a *fractional power series*. It's sufficient to give the first four nonzero terms in the series.

4 Test 2

This test included a note on linearization. This note has been extracted into a seperate section in this archive. See the next section.

Problem 32. Consider the plane autonomous system

$$\frac{dx}{dt} = (1 - y^2)y, \qquad \frac{dy}{dt} = (x^2 - 1)x.$$

1. Find the constant A such that $E = x^4 + y^4 - A\left(x^2 + y^2\right)$ is a first integral.
2. Find all of the equilibrium points.
3. At each equilibrium point (x_0, y_0) compute the Jacobi matrix $J(x_0, y_0)$.
4. Classify the equilibrium points. Be sure to address the stability of each equilibium point.

Problem 33. Consider the system

$$\begin{cases} \dfrac{dx}{dt} = 2 - xy \\ \\ \dfrac{dy}{dt} = 8\,x - y^3. \end{cases}$$

(A) Find all of the equilibrium points. **(B)** Classify the equilibrium points by studying the Jacobi matrix. Be sure to address the stability of each equilibium point.

Problem 34. Consider the system

$$\begin{cases} \dfrac{dx}{dt} = -x - y + x^2 \\ \\ \dfrac{dy}{dt} = x - y. \end{cases}$$

(A) Find all of the equilibrium points. **(B)** Classify the equilibrium points by studying the Jacobi matrix. Be sure to address the stability of each equilibium point.

Problem 35. Consider the linear homogeneous plane system

$$\begin{cases} \dfrac{dx}{dt} = 2\,x + y \\ \\ \dfrac{dy}{dt} = 6\,x - 3\,y. \end{cases}$$

(A) Find the general solution. **(B)** Find the solution with initial values $x(0) = 5$ and $y(0) = -2$.

You may use the eigenvector-eigenvalue method or the elimination method, or any other method that you like.

5 Note on Linearization

Consider the plane autonomous system

$$\begin{cases} \dfrac{dx}{dt} & = f(x, y) \\ \\ \dfrac{dy}{dt} & = g(x, y) \end{cases}$$

where f and g are twice continuously differentiable. Let

$$J(x, y) = \begin{bmatrix} \dfrac{\partial f}{\partial x}(x, y) & \dfrac{\partial f}{\partial y}(x, y) \\ \\ \dfrac{\partial g}{\partial x}(x, y) & \dfrac{\partial g}{\partial y}(x, y) \end{bmatrix}$$

be the Jacobi matrix. Suppose (x_0, y_0) is an isolated nondegenerate critical point of the system. In this case, near (x_0, y_0), we may approximate the solutions of our system by

$$\begin{cases} x(t) & \approx & x_0 + u_1(t) \\ y(t) & \approx & y_0 + u_2(t) \end{cases}$$

where $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a solution of the *linearization* of our system at (x_0, y_0):

$$\frac{d\vec{u}}{dt} = A\vec{u}, \qquad A = J(x_0, y_0).$$

The following table gives the relation between the critical point of the linear system at the origin and the critical point of the original nonlinear system at (x_0, y_0).

Linearization	Linearization	Nonlinear System	Stabililty
distinct real eigenvalues of the same sign	improper node	improper node	asymptotically stable or unstable
real eigenvalue of multiplicity 2, Jacobi matrix not diagonizable	improper node	improper node	asymptotically stable or unstable
real eigenvalue of multiplicity 2, Jacobi matrix diagonizable	proper node	proper node	asymptotically stable or unstable
real eigenvalues, opposite sign	saddle point	saddle point	unstable
complex conjugate eigenvalues, nonzero real part	focus	focus	asymptotically stable or unstable
pure imaginary eigenvalues	center	center	stable
		focus-center	stable
		focus	asymptotically stable or unstable

Note that a *critical point* is the same as an *equilibrium point*.

If the nonlinear system admits a (smooth, nonconstant) first integral in a neighborhood of (x_0, y_0) then (x_0, y_0) is a *center* or a *saddle point*. There are no other possibilities in this case.

6 Test 3

Problem 36. Consider the regular Sturm-Liouville problem

$$\left(x\,y'\right)' + \lambda \frac{1}{x}\,y = 0, \qquad y(1) = 0, \quad y(3) = 0.$$

(A) Find the eigenvalues λ_n and corresponding eigenfunctions $f_n(x)$ for this S-L problem. (Hint: Multiplication by x yields a Cauchy-Euler ordinary differential equation.)

(B) Let $f_m(x)$ and $f_n(x)$ be eigenfunctions with $m \neq n$. What weight function $p(x)$ makes the following orthogonality relationship true?

$$\int_1^3 f_m(x)\, f_n(x)\, p(x)\, dx = 0$$

Problem 37. The Legendre polynomials

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}\left(3\,x^2 - 1\right) \\
P_3(x) &= \frac{1}{2}\left(5\,x^3 - 3x\right) \\
P_4(x) &= \frac{1}{8}\left(35\,x^4 - 30\,x^2 + 3\right) \\
P_5(x) &= \frac{1}{8}\left(63\,x^5 - 70\,x^3 + 15\,x\right) \\
P_6(x) &= \cdots
\end{aligned}$$

are othogonal on $[-1, 1]$ relative to the weight function $p(x) = 1$ and

$$\int_{-1}^{1} |\, P_n(x)\,|^2\ dx = \frac{2}{2\,n + 1}.$$

Let

$$g(x) = \begin{cases} -1 \text{ if } x < 0 \\ 1 \text{ otherwise} \end{cases}$$

and let

$$\sum_{n=0}^{\infty} c_n\, P_n(x)$$

be the expansion of $g(x)$ in a Legendre–Fourier series. Compute the coefficients c_n for $n = 0, 1, \cdots, 5$.

Problem 38. Let A be the matrix

$$\begin{bmatrix} 1 & 9 \\ 4 & 1 \end{bmatrix}.$$

(A) Compute the exponential e^{tA}.

(B) Find the general solution of the linear inhomogeneous system

$$\frac{d\vec{x}}{dt} = A\,\vec{x} + \begin{bmatrix} e^{-5t} \\ e^{3t} \end{bmatrix}$$

where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

7 Contact Information

The contact information below is accurate as of Feb 15, 2001.

Bent E. Petersen	phone numbers
Department of Mathematics	office (541) 737-5163
Oregon State University	home (541) 753-1829
Corvallis, OR 97331-4605	fax (541) 737-0517

bent@alum.mit.edu
petersen@math.orst.edu
http://ucs.orst.edu/~peterseb
http://www.peak.org/~petersen
http://web.orst.edu/~peterseb