Mth 480 Systems of ODE
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Assignment 4 (Modified)

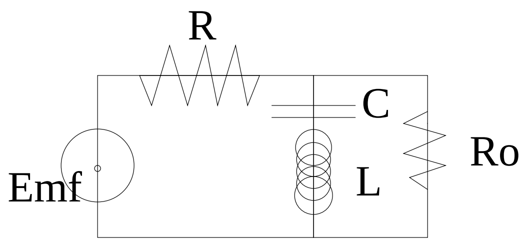
NAME ______________________________

1 Problem. Due May 9.

You may discuss the assigned problems with other people to get ideas (not solutions). The goal, after all is to learn some mathematics, and discussion is an effective way to do so. However, you should do your own write-up and it should be clear, brief and tidy.

Staple this cover sheet onto your submission.

Note: This assignment is an edited version of the original assignment 4.



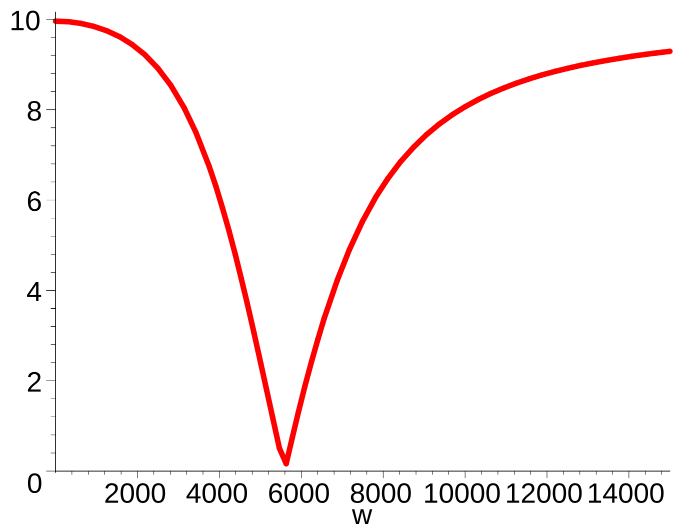
The image above is a (poorly drawn) ciruit diagram. The resistor R_o is the output load. We are interested in computing the voltage drop across it. The object labelled Emf is the input. We assume it is a 10 volt sinusoidal input. By convention we assume the current flows clockwise in the left and right loops. This assumption fixes the sign of the current. Note the capacitor will attenuate low frequency signals and the inductor will attentuate high frequency signals. Thus only middle frequency signals will be shunted directly to ground. Hence we expect the circuit to act as a “band reject” or “notch” filter.

Problem 7. By adding up the voltage drops in each loop we obtain the equations for the system above:

$$\begin{aligned} E &= Ri_1 + \frac{Q}{C} + L\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) \\ 0 &= R_o i_2 - \frac{Q}{C} - L\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) \\ \frac{dQ}{dt} &= i_1 - i_2 \end{aligned}$$

Assume the input signal labeled Emf is $V = 10\cos(\omega t)$ volts, the resistors are $R = 400$ ohms and $R_o = 100000$ ohms, the capacitor is $C = 4\times 10^{-7}$ farads, the inductor is $L = 8\times 10^{-2}$ henrys, the currents i_1 and i_2 are in amperes (coulombs per second), and Q is the charge on the capacitor in coulombs. Find the amplitude of the steady state value i_∞ of the current i_2 and the corresponding voltage drop $V = i_2 R_0$ across the output load R_o as a function of ω.

Here is a plot of the output voltage as a function of ω. Note ω (marked "w" in the plot) is measured in radians per second. To get the frequency in more conventional units, that is, cycles per second, or Hertz (Hz.) divide by 2π.



You may find the calculations quite laborious. Electrical engineers do not actually solve these equations. Instead (as far as I know) they work with the transfer matrix on the Laplace transform side and arrive at the desired conclusions with significantly less work and without actually computing $i_1(t)$, $i_2(t)$ and $Q(t)$.

Addendum

Note that adding up the voltage drops in the outer loop yields

$$E = R i_1 + R_o i_2$$

(0r you can just add the first two equations). Thus if we introduce a new variable $u = i_1 - i_2$ we obtain

$$\begin{aligned} i_1 &= \frac{E + R_o u}{R + R_o} \\ i_2 &= \frac{E - R u}{R + R_o} \\ \frac{dQ}{dt} &= u \end{aligned}$$

.

Substituting in the second and third of the original equations we obtain

$$\begin{aligned}\frac{du}{dt} &= -\frac{R_o R}{(R_o+R)L}u - \frac{1}{LC}Q + \frac{R_o E}{(R_o+R)L} \\ \frac{dQ}{dt} &= u\end{aligned}$$

This inhomogeneous linear system is not too difficult to solve. Finally the output voltage V across the resistor R_o is given by

$$V = i_2 R_o = \frac{E - Ru}{R + R_o} R_o$$

Now toss away the exponentially decaying terms in V and compute the magnitude of the remaining periodic terms as a function of ω.

<u>Comment</u>

What makes this problem so difficult to sort out is that it appears to be third or fourth order, but in fact is really only of second order (total). The other relations are simply algebraic, not differential. If you feed the original equations to Maple then Maple comes up with a solution with just 2 arbitrary parameters! The Laplace transform method doesn't care if you have differential, algbraic or even integral relations (indenfor rimelighedens grænser) and so is probably the better tool for problems of this kind.