Mth 480 Systems of ODE
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Assignment 3

NAME ________________________

3 Problems. Due April 25.

You may discuss the assigned problems with other people to get ideas (not solutions). The goal, after all is to learn some mathematics, and discussion is an effective way to do so. However, you should do your own write-up and it should be clear, brief and tidy.

Staple this cover sheet onto your submission.

Problem 4. Let

$$A(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \tag{1}$$

and consider the system of linear ordinary differential equations given by

$$\frac{dx}{dt} = A(t)x. \tag{2}$$

If we set $z = x_1 + \imath x_2$ then we obtain an equivalent complex scalar ordinary differential equation.

$$\frac{dz}{dt} = \mathrm{e}^{-\imath t} z. \tag{3}$$

Now we can easily solve (3) with initial values $z(0) = \mathrm{e}^{\imath}$ and with $z(0) = \imath\,\mathrm{e}^{\imath}$. The corresponding solutions of (2) are linearly independent over the reals and form successive columns of a fundamental solution matrix $X(t)$ for (2). Find $X(t)$. Then find a Floquet decomposition of $X(t)$. (The last part is completely trivial in the present case. Why?)

Problem 5. Let $A(t)$ be a real $n \times n$ matrix continuous on the real line and periodic with period $T > 0$. Let $X(t)$ be a fundamental solution matrix of the system of ordinary differential equation

$$\frac{dx}{dt} = A(t)x.$$

As we have seen, there is a nonsingular constant matrix E such that

$$X(t+T) = X(t)E.$$

If we choose the $n \times n$ matrix K (possibly complex) such that $E = \mathrm{e}^{TK}$ then we have the Floquet decomposition

$$X(t) = P(t)\,\mathrm{e}^{tK}$$

where $P(t)$ is periodic with period T.

If $\mathrm{tr}(A(t)) = 0$ show that $\det(X(t))$ is constant and therefore $\det(E) = 1$. Conclude $\det(P(t))$ is constant and $\mathrm{tr}(K) = 0$. (Hint: Recall Liouville's theorem: if $W(t) = \det(X(t))$ then $W'(t) = W(t)\,\mathrm{tr}(A(t))$. Here $W(t)$ is the Wronskian.)

Problem 6. Let $X(t)$ and $Y(t)$ be fundamental solution matrices of the Floquet system

$$\frac{dx}{dt} = A(t)\,x$$

where $A(t)$ is continuous and periodic with period T. Suppose we have Floquet representations of solutions

$$\begin{aligned} X(t) &= P(t)\,\mathrm{e}^{tR} \\ Y(t) &= Q(t)\,\mathrm{e}^{tS} \end{aligned}$$

where $P(t)$ and $Q(t)$ are periodic with period T and R and S are constant. Show that $X(t)X(0)^{-1} = Y(t)Y(0)^{-1}$ and therefore

$$X(t+T)X(t)^{-1} = Y(t+T)Y(t)^{-1}$$

for each t. The matrix $M = X(T)X(0)^{-1} = Y(T)Y(0)^{-1}$ is called the monodromy matrix. Show that

$$X(t+T)X(t)^{-1} = P(t)\,\mathrm{e}^{TR}P(t)^{-1}.$$

Conclude that the matrices $X(t+T)X(t)^{-1},\ Y(t+T)Y(t)^{-1},\ X(t)^{-1}X(t+T),\ Y(t)^{-1}Y(t+T),$ $\mathrm{e}^{TR},\ \mathrm{e}^{TS}$ and M all have the same (constant) eigenvalues (indeed, the same Jordan form). These eigenvalues are called the characteristic multipliers. They may be used to determine the stability (or lack of) of the origin.