The Picard Iteration

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Contents

Functional analysis is a branch of mathematics which abstracts many ideas that first arose in differential equations, integral equations, quantum mechanics and calculus of variations. Here we turn the process around and show that the basic existence theorem for differential equations can be obtained from abstract functional analysis arguments, that is, as an application of the contraction mapping principle and as an application of compactness in a function space.

These notes are extracted from a course I gave in functional analysis a few years ago. As a result there may be some (large) gaps in the development. I hope you enjoy the exposition in spite of any roughness.

1 Functional Analysis History

While Fréchet did some early work on linear functionals, described in his thesis important examples of non–normable topological vector spaces and actually introduced a notion of topological affine spaces in the mid 1920's, the modern theory of topological vector spaces dates from the work of Weil [19], [18] in the late 1920's and the work of Kolmogorov [11] and von Neumann in the mid 1930's (see [17]and [13]).

Investigations of integral equations and the needs of quantum mechanics led to the creation of Hilbert space theory and more generally the theory of Banach spaces. Abstract Hilbert space was introduced by von Neumann axiomatically in 1929 though concrete examples of Hilbert spaces had been studied and applied much earlier. Banach in his 1920 thesis introduced an axiomatic theory of normed linear spaces, [1]. Early work on abstract normed spaces was also done by Wiener, Hahn and Helly (see [10]). The very important Hahn–Banach theorem was discovered by Hahn in 1927 and by Banach in 1929. Some earlier less general versions were published by Riesz and Helley. Surprisingly, Banach's 1932 text, [2], the pre–eminent text on normed spaces, still included the axioms for a vector space. Perhaps this notion was not so well–known at the time though Peano had already in 1888 given an axiomatic treatment of abstract vector space, [15] (see also [6] and [13]). One of the characteristics of functional analysis is the study of classes of functionals, in particular the dual space, the space of all continuous linear functionals on a topological vector space. The notion of a continuous functional, that is, a continuous function whose domain consists of functions, was introduced by Volterra in the late 1880's. The word *functional*, well actually *fonctionelle*, was introduced by Hadamard around 1903. The abstract notion of a continuous functional was pioneered by Fréchet. The dual space was introduced in the abstract setting by Hahn in his 1927 paper [8] and subsequently developed by Banach and others.

In summary, the theory of normed spaces has its roots in integral equations and quantum mechanics. Apart from the weak topology, it is largely a metric theory. Banach had no real need for topology in his text, though he had to invent the notion of limits of "transfinite sequences" in order to avoid topology. The basic notions of general topology and topological vector spaces on the other hand arose from the calculus of variations. The non–metric theory of topological vector spaces however languished until the development of Schwartz's theory of distributions in the 1940s. Of course this is a great over–simplification. For a more detailed view of the events briefly sketched above the reader is referred to Morris Kline [10], Jean Dieudonné [7] and Nicola Bourbaki [4].

2 Metric Spaces

The theory of abstract metric spaces was largely created by Fréchet and Hausdorff. Indeed the name *metric space* (metrischer Raum) seems to be due to Hausdorff. The familiar neighborhood formulation of topology is due largely to Hilbert, F. Riesz, Fréchet and especially Hausdorff. Hausdorff's 1914 book [9] became a widely used standard text.

Here we give a brief introduction to the notion of metric space. An inexpensive and useful text with a chapter on metric spaces is Kolmogorov and Fomin, [12].

The three major fundamental existence theorems in the theory of complete metric spaces are the Banach Contraction Mapping Principle, the Baire Category Theorem, and the characterization of compact sets in terms of total boundedness. We will examine only the contraction mapping principle in these notes (see the next section).

Let X be a set. A *semimetric* (also called a *quasimetric*) on X is a function

$$d : X \times X \to \mathbb{R}$$

such that

1. $d(x,y) \geq 0, \quad d(x,x) = 0$
2. $d(x,y) = d(y,x)$
3. $d(x,y) \leq d(x,z) + d(z,y)$

for each $x, y, z \in X$. If in addition we require $d(x,y) = 0 \Rightarrow x = y$ then d is called a *metric*. A *semimetric space* (respectively, a *metric space*) is a set X together with a semimetric (respectively, a metric) d on X. Most of the results obtained for metric spaces apply equally well to semimetric spaces, though some care is needed, as, for example, a compact set in a semimetric space need not be closed, limits of convergent sequences need not be unique, and so forth.

The most familiar example of a metric space is the set of real numbers $\mathbb{R}$ with the usual metric

$$d_1(x,y) = |x - y|, \quad x, y \in \mathbb{R}.$$

Another useful metric on $\mathbb{R}$ is given by

$$d_2(x,y) = |\arctan(x) - \arctan(y)|.$$

Given $x \in X$ and $\varepsilon > 0$ we define the *open ε–ball at x* to be

$$W(x,\varepsilon) = \{ y \in X \mid d(x,y) < \varepsilon \}.$$

A subset U of X is said to *open* if for each $x \in U$ there is $\varepsilon > 0$ such that $W(x,\varepsilon) \subseteq U$. Note that the empty set $\varnothing$ and the whole space X are open.

A subset V of X is said to be a *neighborhood* of x if $W(x,\varepsilon) \subseteq V$ for some $\varepsilon > 0$.

A subset A of X is *closed* if its complement is open. An example of a closed set is the *closed ε–ball at x*

$$B(x,\varepsilon) = \{ y \in X \mid d(x,y) \leq \varepsilon \}.$$

The *closure* of a subset A of X is the intersection of all closed supersets of A. It is easy to see that the closure of A is closed.

The *interior* of a subset A of X is the union of all open subsets of A. It is easy to see that the interior of A is open.

Exercise 2.1. *Show that $B(x,\varepsilon)$ is closed. Give an example where the closure of $W(x,\varepsilon)$ is a proper subset of $B(x,\varepsilon)$. Show that $W(x,\varepsilon)$ is open.*

Two important notions introduced early in the study of metric spaces are convergence of sequences and continuity of functions. Initially these concepts are given a metric formulation, but they are usually quickly rephrased in terms of neighborhoods and open sets.

Let $(x_n)_{n\geq 1}$ be a sequence in the metric space X. We say that $(x_n)_{n\geq 1}$ *converges* to $x \in X$ and we write

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x$$

if for each $\varepsilon > 0$ there is N such that

$$d(x,x_n) < \varepsilon \text{ if } n \geq N.$$

We can formulate the notion of convergence entirely in terms of neighborhoods with no direct reference to the metric as follows: $x_n \to x$ if and only if for each neighborhood V of x there exists N such that $x_n \in V$ for each $n \geq N$.

Let X and Y be metric spaces and let $f : X \to Y$. Let $a \in X$. We say that f is *continuous at a* if for each $\varepsilon > 0$ there is $\delta > 0$ such that $x \in X$, $d(x,a) < \delta$ implies $d(f(x), f(a)) < \varepsilon$. We say f is *continuous on X* or simply *continuous* if f is continuous at each point of X.

Another important notion in metric space is uniform continuity. Let X and Y be metric spaces and let $f : X \to Y$. We say f is *uniformly continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$.

An *isometry* is a function $f : X \to Y$ such that $d(f(x), f(y)) = d(x,y)$ for each $x,y \in X$. Clearly an isometry is uniformly continuous.

Let X be a metric space with metric d. A sequence $(x_n)_{n\geq 1}$ in X is said to be a *Cauchy sequence* if for each $\varepsilon > 0$ there exists N such that $n \geq N$ and $m \geq N$

implies $d(x_n, x_m) < \varepsilon$. Obviously a convergent sequence is Cauchy. Given any metric space X there is a complete metric space Y such that X is isometric to a dense subset of Y. One popular proof is to define Y in terms of equivalence classes of Cauchy sequences in X. See if you can work out an argument, or see [12], chapter 2, section 7, theorem 4. The space Y is essentially unique and is called the *completion* of X.

Exercise 2.2. *If the Cauchy sequence $(x_n)_{n\geq 1}$ has a convergent subsequence then it converges.*

Let X be a metric space. A subset $A \subset X$ is said to be *complete* if each Cauchy sequence in A is convergent to a point in A.

Exercise 2.3. *Let X be a metric space. Show each complete subset of X is closed. If X is complete show each closed subset is complete.*

Exercise 2.4. *We know $(\mathbb{R}, d_1)$ is complete. Show that $(\mathbb{R}, d_2)$ is not complete. (The metrics d_1 and d_2 are defined above.)*

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Exercise 2.6. *Let X be a metric space. Show each complete subset of X is closed. If X is complete show each closed subset is complete.*

Exercise 2.7. *We know $(\mathbb{R}, d_1)$ is complete. Show that $(\mathbb{R}, d_2)$ is not complete. (The metrics d_1 and d_2 are defined above.)*

3 Contraction Mapping Principle

The Contraction Mapping Principle was formulated by Banach in his 1920 thesis, published as [1] in 1922 (see page 160). The proof that Banach gave is the same as the one we give below, except that Banach works in a normed space and makes use of the linear structure.

There is a nice discussion of the Contraction Mapping Principle, and some applications, in Kolmogorov and Fomin, [12].

Given a function $f:X \to X$ a *fixed point* of f is a point $x \in X$ such that $f(x) = x$. A well-known topological theorem asserting the existence of at least one fixed point is the Brouwer theorem,

Theorem 3.1 (Brouwer). *If B is a closed ball in $\mathbb{R}^n$ and $f:B \to B$ is a continuous map then f has a fixed point in B.*

Exercise 3.1. *Prove the Brouwer theorem in dimension 1, that is show if $f: [a,b] \to [a,b]$ is a continuous function then f has a fixed point.* **Hint:** *Consider the function g defined by $g(x) = x - f(x)$ and by looking at sign changes show that g must have a root in $[a,b]$.*

Since the Brouwer theorem deals with a finite dimensional situation it is not of much importance in functional analysis. However, the *Contraction Mapping Principle* which we study in the present section applies in many function spaces. In particular, as we will see, it implies the initial value problem for differential equations, under mild hypotheses, has a unique solution.

Let X be a metric space. A mapping $T:X \to X$ is called *contraction map* if there exists a constant c with $0 \le c < 1$ such that

$$d(T(x), T(y)) \le c\, d(x,y), \quad x,y \in X.$$

The constant c is called the *contractivity coefficient.*

Theorem 3.2 (Contraction Mapping Principle). *Let X be a complete metric space and let $T:X \to X$ be a contraction mapping with contractivity coefficient c. Let $x_0 \in X$ and inductively define*

$$x_{n+1} = T(x_n), \quad n \ge 0.$$

The T has a unique fixed point a, the sequence x_n converges to a and

$$d(a,x_n) \le c^n d(a,x_0).$$

Proof. If $n \ge 1$ then $d(x_n,x_{n+1}) = d(T(x_{n-1}, T(x_n)) \le c\, d(x_{n-1},x_n)$ and so by induction $d(x_n,x_{n+1}) \le c^n d(x_0,x_1)$. It follows if $0 \le n < m$ then

$$d(x_n,x_m) \le \left(c^n + \cdots + c^{m-1}\right) d(x_0,x_1) \le \frac{c^n}{1-c} d(x_0,x_1)$$

and therefore (x_n) is a Cauchy sequence. Since X is complete this sequence converges to a point a. Now $T(a) = a$ by continuity of T.

If a is a fixed point of T, that is, $T(a) = a$ then $d(a,x_n) = d(T(a), T(x_{n-1}) \le c\, d(a,x_{n-1})$. By induction we obtain $d(a,x_n) \le c^n d(a,x_0)$.

For uniqueness note if a and b are fixed points then $d(a,b) = d(T(a),T(b)) \le c\, d(a,b)$ which implies $d(a,b) = 0$ since $0 \le c < 1$. □

Note taking the limit as $m \to \infty$ in the inequality in the proof of the theorem we obtain

$$d(a,x_n) \le \frac{c^n}{1-c} d(x_0,x_1).$$

This inequality gives us a very explicit estimate of the error in x_n when viewed as an estimate of a. It is sometimes very convenient.

The contraction mapping principle is useful, but the hypotheses are very strong and may be difficult to satisfy. If we examine the proof of the contraction mapping principle we are led to a simple result which may be more useful.

Theorem 3.3 (Picard iteration). *Let X be a complete metric space and let $T : X \to X$ be a continuous map. Let $x_0 \in X$ and inductively define*

$$x_{n+1} = T(x_n).$$

If

$$\sum_{n=0}^{\infty} d(x_n,x_{n+1}) < \infty$$

then the sequence $(x_n)_{n\ge 0}$ converges to a fixed point a of T. Moreover

$$d(a,x_n) \le \sum_{k=n}^{\infty} d(x_k,x_{k+1}).$$

Proof. By the triangle inequality if $0 \le n < m$ then

$$d(x_n,x_m) \le \sum_{k=n}^{m-1} d(x_k,x_{k+1}).$$

Thus $(x_n)_{n\ge 0}$ is a Cauchy sequence. If the limit is a then by continuity of T we have $a = T(a)$. Taking the limit as $m \to \infty$ in the inequality above we obtain the required estimate. □

4 Example: Iterated Function Systems

In this section we describe an application of Banach's Contraction Mapping Principle, [1], to the study of iterated function systems and their corresponding fractals. For a large number of examples and many of the details not presented here see the book by M. Barnsley, [3].

Let X be a metric space. If B is a nonempty subset of X we define

$$d(x,B) = \inf_{y\in B} d(x,y).$$

Let $x, y \in X$. If $\varepsilon > 0$ choose $z \in B$ such that $d(y,z) < d(y,B) + \varepsilon$. Then $d(x,B) \leq d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + d(y,B) + \varepsilon$. Thus we conclude

$$|d(x,B) - d(y,B)| \leq d(x,y), \quad x, y \in X.$$

It follows that $x \longrightarrow d(x,B)$ is a continuous nonnegative function vanishing precisely on $\overline{B}$.

Now if A and B are nonempty subsets of X we define

$$d(A,B) = \sup_{x \in A} d(x,B).$$

Obviously $0 \leq d(A,B) \leq \infty$. Let A, B, C be nonempty subsets of X. Let $x \in A$ and let $\varepsilon > 0$. Choose $z \in C$ such that $d(x,z) < d(x,C) + \varepsilon$. Now $d(x,B) \leq d(x,z) + d(z,B) \leq d(x,C) + d(z,B) + \varepsilon$. Since $z \in C$ we have $d(z,B) \leq d(C,B)$ and therefore $d(x,B) \leq d(x,C) + d(C,B)$. Taking the supremum over $x \in A$ we obtain

$$d(A,B) \leq d(A,C) + d(C,B).$$

We have to be careful with the order here:

Exercise 4.1. *Find an example with* $d(A,B) \neq d(B,A)$.

Let $\mathfrak{K}(X)$ be the set of nonempty compact subsets of X. If $A, B \in \mathfrak{K}(X)$ we define the *Hausdorff metric* d_H by

$$d_H(A,B) = \max\left\{d(A,B), d(B,A)\right\}.$$

If A is a subset of X we define the *closed* ε*–neighborhood* A_ε of A by

$$A_\varepsilon = \{y \in X \mid d(x,y) \leq \varepsilon \text{ for some } x \in A\}.$$

Exercise 4.2. *If* $A, B \in \mathfrak{K}(X)$ *and* $\varepsilon > 0$ *then* $d(A,B) \leq \varepsilon$ *if and only if* $A \subseteq B_\varepsilon$.

Exercise 4.3. d_H *is a metric on* $\mathfrak{K}(X)$.

Theorem 4.1. *Completeness of* $\mathfrak{K}(X)$. *If* X *is a complete metric space then* $\mathfrak{K}(X)$ *is complete in the Hausdorff metric. If* $(A_n)_{n\geq 1}$ *is a Cauchy sequence in* $\mathfrak{K}(X)$ *and* $A = \lim_{n\to\infty} A_n$ *then*

$$A = \left\{x \in X \mid \text{there exists } x_n \in A_n \text{ so that } \lim_{n\to\infty} x_n = x\right\}.$$

For a proof of this theorem see [3].

Now let T_j be contraction mappings of X with contractivity coefficients c_j, $j = 1, \cdots n$. Define $T : \mathfrak{K}(X) \to \mathfrak{K}(X)$ by

$$T(A) = T_1(A) \bigcup \cdots \bigcup T_n(A), \quad A \in \mathfrak{K}(X).$$

One can show that T is a contraction mapping with contractivity coefficient $c = \max_j c_j$. Thus T has a unique fixed point A in $\mathfrak{K}(X)$ and we can approximate it by computing iterates of T applied to an arbitrary element of $\mathfrak{K}(X)$. The system $(T_j)_{1\leq j\leq n}$ is called an *iterated function system*, abbreviated as IFS. The fixed point A is called the *fractal* determined by the IFS.

Exercise 4.4. *Let $X = [0,1]$ and consider the IFS $\{T_1, T_2\}$ defined by $T_1(t) = t/3$ and $T_2(t) = 2/3 + t/3$. Show by a direct calculation that the Cantor set is the fixed point of this IFS.*

5 Example: Ordinary Differential Equations

In this section we show how Contraction Mapping Principle implies a version of the fundamental existence and uniqueness theorem for Cauchy initial value problems for ordinary differential equations.

This application of the Contraction Mapping Principle is also discussed in Kolmogorov and Fomin [12], chapter 2, section 8, where there is also an application to solving Fredholm integral equations of the second kind and Volterra integral equations.

Let Ω be an open subset of the plane and let f be a continuous function in Ω. Let $(a,c) \in \Omega$ and consider the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(a) = c.$$

By integrating we see this initial value problem is equivalent to the integral equation

$$y(x) = c + \int_a^x f(t, y(t))\, dt,$$

that is, we are looking for a fixed point for the map defined by the integral. Let $p > 0$ and $q > 0$ be chosen so that the rectangle D defined by

$$0 \le x - a \le p, \quad |y - c| \le q$$

is contained in Ω. Let M be an upper bound for f on the rectangle D. Choose h with

$$0 < h \le \min(p, q/M).$$

Let $b = a + h$ and let

$$X_h = \{\, u \in \mathrm{C}([a,b]) \mid u(a) = c, \quad |u(x) - c| \le q \text{ if } a \le x \le b \,\}.$$

If $u \in X_h$ define Tu by

$$Tu(x) = c + \int_a^x f(t, u(t))\, dt.$$

Clearly

$$|Tu(x) - c| \le M(x-a) \le Mh \le q$$

and so $T : X_h \to X_h$. The space $\mathrm{C}([a,b])$ is a complete metric space when provided with the metric $d(u,v) = \sup |u(t) - v(t)|$, $a \le t \le b$ and X_h is a closed subset. Thus

X_h is a complete metric space. Now suppose that f is Lipschitz continuous in the second variable in the rectangle D, say

$$|\, f(x,y) - f(x,z)\,| \le L\,|y - z\,|, \qquad \text{for } (x,y),(x,z) \in D.$$

Let $0 < \delta < 1$. If $u,v \in X_h$ then

$$\begin{aligned} |\,Tu(x) - Tv(x)\,| &\le \int_a^x |\, f(t,u(t)) - f(t,v(t))\,|\,dt \\ &\le \int_a^x L\,|\,u(t) - v(t)\,|\,dt \le Lh\,d(u,v). \end{aligned}$$

Thus T is continuous. Moreover if $Lh \le \delta$ then this inequality implies that T is a contraction mapping and so has a unique fixed point in X_h. A simple estimate shows that any solution of the initial value problem must be in X_h. Thus we have the existence of a unique solution to the initial value problem on the interval $[a,b]$ where $b = a + h$ and

$$h = \min(p, q/M, \delta/L).$$

The iterates $u_{n+1} = T(u_n)$ converge rapidly to the fixed point. This method of approximating the solution was published by Liouville as early as 1838. The general method is usually credited to Picard, 1890 (see [16]). The successive approximants are usually called *Picard iterates*.

We can do better by using the slightly strengthened Contraction Mapping Principle, which I referred to as the *Picard iteration theorem* earlier.

Let $y_0(x) = c$ and inductively define $y_{n+1} = Ty_n$. Then

$$|\,y_1(x) - y_0(x)\,| \le \int_a^x |\, f(t,y_0(t))\,|\,dt \le M\,(x-a)$$

and

$$\begin{aligned} |\,y_2(x) - y_1(x)\,| = |\,Ty_1(x) - Ty_0(x)\,| &\le \int_a^x L\,|\,y_1(t) - y_0(t)\,|\,dt \\ &\le \int_a^x LM\,(t-a)\,dt = \frac{LM(x-a)^2}{2}. \end{aligned}$$

By induction we see

$$|\,y_n(x) - y_{n-1}(x)\,| \le \frac{L^{n-1}M(x-a)^n}{n!}$$

and so

$$d(y_n, y_{n-1}) \le \frac{M}{L}\,\frac{(Lh)^n}{n!}.$$

It follows that the sequence $(y_n)_{n\ge 0}$ converges to a fixed point of T. Thus we have existence of a solution to the initial value problem on the interval $[a,b]$ where $b = a + h$ and

$$h = \min(p, q/M).$$

Uniqueness requires an additional argument. See, for example, Nemytskii and Stepanov [14] or Coddington and Levinson [5]. Finally note that the argument is essentially unchanged if y and f are vector valued. Since any system of differential equations can be replaced by a system of first order equations, we obtain existence and uniqueness under mild hypotheses for the Cauchy initial value problem for any system of differential equations (in normal form).

6 Existence without uniqueness

This section is very sketchy since a proper development requires substantial technical analysis. In particular, the Ascoli-Arzelà lemma is required.

Consider the initial value problem

$$\begin{cases} \frac{dy}{dx} &= f(x,y) \\ y(x_0) &= y_0. \end{cases}$$

where f is continuous. We replace the problem with an equivalent integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

This equation we attempt to solve by the Picard-Liouville iterative scheme

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\, dt$$

where we take $y_0(x)$ to be the constant function y_0. If $|f| \le M$ and f is Lipschitz continuous in the second variable with Lipschitz constant L we obtain easily the inequality

$$|y_{n+1}(x) - y_n(x)| \le ML^n \frac{|x-x_0|^{n+1}}{(n+1)!}$$

which implies implies convergence to a solution, as we have seen in a previous section. We can also show that the solution is unique.

Suppose now, however that we just have f bounded by M and continuous. Then we have the estimates

$$|y_n(x)| \le |y_0| + M|x - x_0|$$

$$|y_n(x) - y_n(z)| \le M|x - z|.$$

Thus the sequence (y_n) is pointwise bounded and equicontinuous. By the Ascoli-Arzelà lemma a subsequence converges uniformly on compacta. Again we obtain a solution to the initial value problem, though this time it need not be unique. The bound M on $|f|$ above is an unnecessary hypothesis as it follows locally from the continuity. Thus only continuity is necessary to obtain existence of solutions to the ivp.

Note we used the fact that compactness in $C(\mathbb{R})$ implies sequential compactness. This fact follows from the fact that $C(\mathbb{R})$ is metrizable.

Instead of using the Picard-Liouville iteration on can use Euler's scheme – construct piecewise linear approximate solutions. Again one can prove the appropriate bounds for pointwise boundedness and equicontinuity and then apply Ascoli-Arzelà to get a solution. See for example [14] or [5].

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