Mth 480 Systems of ODE
Bent E. Petersen 20060524
Assignment 5

NAME ______________________________

7 Problems. Due May 31.

You may discuss the assigned problems with other people to get ideas (not solutions). The goal, after all is to learn some mathematics, and discussion is an effective way to do so. However, you should do your own write-up and it should be clear, brief and tidy.

Staple this cover sheet onto your submission.

Consider the plane autonomous system

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned}$$

where f and g are twice continuously differentiable. Let

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{pmatrix}$$

be the Jacobi matrix. Suppose (x_0,y_0) is an isolated critical point of the system, that is, $f(x_0,y_0) = 0,\ \ g(x_0,y_0) = 0$. Suppose in addition this critical point is not degenerate, that is $\det J(x_0,y_0) \neq 0$. In this case we may write the solutions of our system in the form

$$\begin{aligned}x(t) &= x_0 + u_1(t)\\ y(t) &= y_0 + u_2(t)\end{aligned}$$

where

$$\frac{du}{dt} = Au + R(u), \quad A = J(x_0,y_0) \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Here $R(u)$ consists of the remainder terms from the Taylor expansions of f and g. For small $|u|$ we have $|Au| \geq c|u|$ and $|R(u)| \leq c|u|^2$. Hence we expect that we can neglect $R(u)$. Thus we are led to the *linearization* of our system at (x_0,y_0)

$$\frac{du}{dt} = Au.$$

We may study this linear system by considering the various Jordan canonical forms of A. The following table gives the relation between the critical point of the linear system at the origin and the critical point of the original nonlinear system at (x_0,y_0).

Linearization	Linearization	Nonlinear System	Stabililty
distinct real eigenvalues of the same sign	improper node	improper node	asymptotically stable or unstable
real eigenvalue of multiplicity 2, Jacobi matrix not diagonizable	improper node	improper node	asymptotically stable or unstable
real eigenvalue of multiplicity 2, Jacobi matrix diagonizable	proper node	proper node	asymptotically stable or unstable
real eigenvalues, opposite sign	saddle point	saddle point	unstable
complex conjugate eigenvalues, nonzero real part	focus	focus	asymptotically stable or unstable
pure imaginary eigenvalues	center	center	stable
		focus-center	stable
		focus	asymptotically stable or unstable

If there is a neighborhood U of (x_0, y_0) and a function h continuously differentiable in U such that

$$f\frac{\partial h}{\partial x} + g\frac{\partial h}{\partial y} = 0 \text{ in } U$$

and

$$\left|\frac{\partial h}{\partial x}\right| + \left|\frac{\partial h}{\partial y}\right| \neq 0 \text{ in } U \text{ except possibly at } (x_0, y_0),$$

then (x_0, y_0) is a center or a saddle point for the nonlinear system. (This is obvious.) The function h is called a *first integral*. Physically it corresponds to a conserved quantity and is easily seen to be constant on each orbit.

The critical point (x_0, y_0) is said to be hyperbolic if all of the eigenvalues of A have nonzero real part. In our case since $\det(A) \neq 0$ nonhyperbolic simply means the eigenvalues are both pure imaginary (since they occur in conjugate pairs). The fact, hinted at in the table, that the solutions of the original system (near the critical point) and the solutions of the linearization, in the hyperbolic case, are diffeomorphic is known as Hartman's theorem.

We required f and g to be twice continuously differentiable above. If we require f and g to be merely continuously differentiable the situation is more complicated. For example, it is possible for the nonlinear system to have a focus at a whereas the linear system has a proper node at the origin. (Due to Perron - see our text page 142).

Here are a few problems for you.

Problem 8. Find the four critical points of the system

$$\begin{aligned} \frac{dx}{dt} &= y^2 - 1 \\ \frac{dy}{dt} &= 1 - x^2 \end{aligned}$$

and classify them. Note $x+y-\frac{1}{3}(x^3+y^3)$ is a first integral.

Problem 9. Find the five critical points of the system

$$\begin{aligned}\frac{dx}{dt} &= (1-y^2)x \\ \frac{dy}{dt} &= (x^2-1)y\end{aligned}$$

and classify them. Note $x^2+y^2-\log(x^2y^2)$ is a first integral.

Problem 10. Find the nine critical points of the system

$$\begin{aligned}\frac{dx}{dt} &= (1-y^2)y \\ \frac{dy}{dt} &= (x^2-1)x\end{aligned}$$

and classify them. Note $2(y^2-x^2)+(x^4-y^4)$ is a first integral.

Problem 11. Find the two critical points of the system

$$\begin{aligned}\frac{dx}{dt} &= 2-xy \\ \frac{dy}{dt} &= 8x-y^3\end{aligned}$$

and classify them.

Problem 12. Find the two critical points of the system

$$\begin{aligned}\frac{dx}{dt} &= x+xy \\ \frac{dy}{dt} &= 2y-xy\end{aligned}$$

and classify them.

Problem 13. Find the two critical points of the system

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x(1-x)+3y\end{aligned}$$

and classify them.

Problem 14. Find the two critical points of the system

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -2x(1-x)+y\end{aligned}$$

and classify them.