Newton's Method and the Secant Method

Mth 351 Oct 3 2001 Maple 6

Bent E. Petersen

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The four problems below constitute Assignment 1. Your solutions (carefully written) are due Oct 10.

Newton's method and the secant method are just iterative schemes for the functions GN and GS, respectively, where

```
> GN:=(f,x)->x-f(x)/(D(f)(x));
```

$$GN := (f, x) \to x - \frac{f(x)}{\mathrm{D}(f)(x)}$$

```
> GS:=(f,x,xp)->x-f(x)*(x-xp)/(f(x)-f(xp));
```

$$GS := (f, x, xp) \to x - \frac{f(x)\,(x - xp)}{f(x) - f(xp)}$$

Note with these definitions Maple will attempt symbolic evaluations. Even for very simple functions Maple will rapidly grind to a halt. Try to force floating point evaluation to keep things reasonable. For example write the initial points in floating point format, that is 1.0 rather than 1, so Maple will use floating point arithmetic at Digits precision.

```
> f:=x->x^2-2;
```

$$f := x \to x^2 - 2$$

```
> Digits:=36: GN(f,1.0); for k from 1 to 5 do GN(f,%); od;
```

$$1.500000000000000000000000000000000000$$
$$1.416666666666666666666666666666666667$$
$$1.414215686274509803921568627450980392$$
$$1.414213562374689910626295578890134910$$
$$1.414213562373095048801689623502530250$$
$$1.414213562373095048801688724209698080$$

```
> a[0]:=0: a[1]:=1.0: for k from 2 to 14 do
  a[k]:=GS(f,a[k-1],a[k-2]); od;
```

$$a_2 := 2.000000000000000000000000000000000000$$
$$a_3 := 1.333333333333333333333333333333333333$$
$$a_4 := 1.400000000000000000000000000000000000$$

$$a_5 := 1.41463414634146341463414634146341463$$

$$a_6 := 1.41421143847487001733102253032928943$$

$$a_7 := 1.41421356205732046262813424583843878$$

$$a_8 := 1.41421356237309528592060931000112016$$

$$a_9 := 1.41421356237309504880168869773698879$$

$$a_{10} := 1.41421356237309504880168872420969808$$

$$a_{11} := 1.41421356237309504880168872420969808$$

$$a_{12} := \text{Float(undefined)}$$

$$a_{13} := \text{Float(undefined)}$$

$$a_{14} := \text{Float(undefined)}$$

What happened? The successive iterates are so close that we are trying to evaluate 0/0 (within the specified precision). Let's increase the precision and try again.

```
> Digits:=69:a[0]:=0: a[1]:=1.0: for k from 2 to 14 do
  a[k]:=GS(f,a[k-1],a[k-2]); od;
```

$$a_2 := 2.00000000000000000000000000000000000000000000000000000000000000000000$$

$$a_3 := 1.33333333333333333333333333333333333333333333333333333333333333333333$$

$$a_4 := 1.40000000000000000000000000000000000000000000000000000000000000000000$$

$$a_5 := 1.41463414634146341463414634146341463414634146341463414634146341463415$$

$$a_6 := 1.41421143847487001733102253032928942807625649913344887348353552859619$$

$$a_7 := 1.41421356205732046262813424583843877947461223513941603657989898688025$$

$$a_8 := 1.41421356237309528592060931000112015748201929326138566282292906879717$$

$$a_9 := 1.41421356237309504880168869773698878542263607534134734941257550225515$$

$$a_{10} := 1.41421356237309504880168872420969807856966965605858641103592235922528$$

$$a_{11} := 1.41421356237309504880168872420969807856967187537694807317667973799075$$

$$a_{12} := 1.41421356237309504880168872420969807856967187537694807317667973799073$$

$$a_{13} := 1.41421356237309504880168872420969807856967187537694807317667973799073$$

$$a_{14} := \text{Float(undefined)}$$

In this example (and generally) the secant method converges more slowly than Newton's method, but still converges very rapidly indeed. For simple roots Newton's method is quadratic and so roughly doubles the number of significant figures at each iteration (after a while). The secant method has exponent roughly 1.6 and so multiplies the number of significant figures at each step (after a while) by about 1.6 .

```
> Digits:=24:
```

Consider now p(x) = $x^4 - 5\,x^3 + 7\,x^2 - 15\,x + 36$ which has a double root at 3. With initial guess 2.0 Newton's method yields

```
> p:=x->x^4-5*x^3+7*x^2-15*x+36;
```

$$p := x \to x^4 - 5\,x^3 + 7\,x^2 - 15\,x + 36$$

```
> GN(p,2.0); for k from 1 to 6 do GN(p,%); od;
```

2.66666666666666666667
2.84716157205240174672492
2.92628777820071775174780
2.96375479813927815200109
2.98202303887463716141612
2.99104710308510535885771
2.99553234759223455588036

This is much slower convergence than in the case of a single root - in fact, the convergence is only linear. If we use the modified Newton's method, GN2() we will have quadratic convergence again.

```
> GN2:=(f,x)->x-2*f(x)/(D(f)(x));
```

$$GN2 := (f, x) \to x - \frac{2\,f(x)}{\mathrm{D}(f)(x)}$$

```
> GN2(p,2.0); for k from 1 to 6 do GN2(p,%); od;
```

3.33333333333333333333
3.02159624413145539906099
3.00010121368241457398813
3.00000000224083679067196
3.00000000000001652499334
3.00000000037823150350210
2.99999999999997440794080

In like manner we have modified Newton methods GNk() where k is an integer greater than or equal to 1 and GNk() is defined in the obvious way. We then have quadratic convergence in a neighborhood of a root of multiplicity k. Unfortunately, we usually have only linear convergence near roots of multiplicities other than k

Problem 1. Investigate (numerically) the behavior of GS2() near a double root and near a simple root. Here

```
> GS2:=(f,x,xp)->x-2*f(x)*(x-xp)/(f(x)-f(xp));
```

$$GS2 := (f, x, xp) \to x - \frac{2 f(x) (x - xp)}{f(x) - f(xp)}$$

Apply GS2() to the polynomial p(x) defined above. Also apply GS(). Compare the results.

Problem 2. Apply Newton's method to

```
> p:=x->x^2-x-1;
```

$$p := x \to x^2 - x - 1$$

to estimate

```
> (sqrt(5)+1)/2: %=evalf(%,40);
```

$$\frac{1}{2}\sqrt{5} + \frac{1}{2} = 1.6180339887498948482045868343656381177 20$$

Problem 3. Show f(x) = x + cos(x) has a unique root. Show the root lies in the interval from $-\frac{\pi}{2}$ to 0. Compare the bisection method, Newton's method (initial guess 0) and the secant method (initial guesses $-\frac{\pi}{2}$ and 0) for estimating this root. According to Maple the root is about

```
> Digits:=40: alpha = fsolve(x+cos(x)=0,x); Digits:=10:
```

$$\alpha = -.7390851332151606416553120876738734040134$$

For Newton's method and the secant method compute the actual error $|\alpha - x_n|$ at each step and compare it with $|x_{n+1} - x_n|$.

Problem 4. Apply Newton's method with initial guess 0.5 to the function

```
> h:=x->1-1/(1+x^2);
```

$$h := x \to 1 - \frac{1}{1 + x^2}$$

Why is the convergence so slow?

```
>
```