Linear Algebra – Mth 341

Archive – Summer 1997 Files

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This archive contains the sample problems and tests from Mth 341 Summer 1997. The original test instructions, headers and formatting have not been preserved.

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1 Sample Problems

Problem 1. Let A be the matrix

$$A = \begin{bmatrix} t & 4 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find all values of t such that the row reduced echelon form (*aka*, Gauss–Jordan completely reduced row echelon canonical form) of A has at least one zero row.

Problem 2. let

$$A = \begin{bmatrix} 1 & -1 & 5 & 1 \\ -1 & 1 & -7 & -1 \\ 2 & 0 & 4 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Find condition(s) on b (if any) which guarantee a solution, and then write the set of all solutions in the vector form (*aka*, canonical parametric vector form).

Problem 3. Let

$$A = \begin{bmatrix} 0 & 2 & 1 & 1 & 3 \\ 2 & 6 & 5 & 7 & 7 \\ 4 & 14 & 9 & 15 & 15 \\ 8 & 28 & 11 & 30 & 23 \\ 6 & 20 & 14 & 22 & 22 \end{bmatrix}.$$

Do the rows of A span $\mathbb{R}^5$? Do the columns of A span $\mathbb{R}^5$?

Problem 4. Let

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 2 \\ 1 & 4 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Compute the matrix $C = AB - BA$.

Problem 5. Let

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 6 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix}.$$

Find a nontrivial solution of the system of linear eqautions

$$Ax = 10\,x$$

if there is one. Otherwise prove there is no nontrivial solution.

Problem 6. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map. Suppose

$$T(e_1 + e_2) = e_1 - e_3, \quad T(e_1 - e_2) = e_2 + e_3.$$

Find the matrix of T (with respect to the canonical bases). Is T one-to-one? Is T onto?

Problem 7. Let A be an $m \times n$ matrix with $m < n$. Prove that the columns of A are linearly dependent.

Problem 8. Let A be an $m \times n$ matrix. Prove that the columns of A span $\mathbb{R}^m$ if and only if A has $n - m$ non-pivotal (or free) columns.

Problem 9. Prove that a set consisting of two vectors is linearly dependent if and only if at least one of the vectors is a scalar multiple of the other one.

Problem 10. Let v_1, v_2, v_3 be linearly independent vectors in $\mathbb{R}^n$. Let $w_1 = v_2 + v_3$, $w_2 = v_3 + v_1$ and $w_3 = v_1 + v_2$. Prove or disprove that the set $\{w_1, w_2, w_3\}$ is linearly independent.

Problem 11. Let M be the augmented matrix of a system of m linear equations in n unknowns. How many rows does M have? How many columsn does M have? If the last column of M is pivotal is the system consistent or inconsistent?

Problem 12. Let

$$M = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 5 & 6 & 7 & 6 \\ 9 & 10 & 11 & h \end{bmatrix}$$

be the augmented matrix of a system of linear equations. For what value(s) of h is the system inconsistent?

Problem 13. Let M be the augmented matrix of a system of m linear equations in n unknowns. Consider the following statements:

1. n columns of M are pivotal

2. the first n columns of M are pivotal
3. the last column of M is pivotal
4. M has fewer than n pivotal columns
5. M has fewer than n pivotal columns and the last column is not pivotal
6. M has $n+1$ pivotal columns
7. M has m rows which contain pivots
8. M has m rows which contain pivots among the first n columns
9. fewer than m rows of M contain pivots
10. fewer than m rows of M contain pivots and the last column is not pivotal

For each of the following statements indicate which of the above statements, if any, are exactly equivalent to it.

1. the system has a unique solution
2. the system has several solutions
3. the system has no solutions

2 Test 1

Problem 14. Suppose

$$M = \begin{bmatrix} 0 & 2 & 0 & 1 & -1 & -2 \\ 4 & -4 & 3 & -4 & -4 & 20 \\ 1 & 2 & 0 & 2 & 5 & -7 \\ 2 & 5 & 1 & 2 & -3 & -1 \end{bmatrix}$$

is the augmented matrix of a system of linear equations. After a lengthy calculation we find the row reduced echelon form (*aka*, Gauss–Jordan completely reduced row echelon canonical form) R of M is given by

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 & -4 \end{bmatrix}.$$

Part (A): How many linear equations are there in the original system?

Part (B): How many variables?

Part (C): Which variables are free?

Part (D): Which variables are pivotal (bound)?

Part (E): If the system is consistent write the solution in vector form (*aka*, canonical parametric vector form).

Problem 15. Let A be a 4×5 matrix, let b be a 4×1 matrix, and let M be the augmented matrix $M = [A, b]$. Suppose

$$M = \begin{bmatrix} 0 & 2 & 0 & 1 & -1 & -2 \\ 4 & -4 & 0 & -4 & -4 & 20 \\ 1 & 2 & 0 & 2 & 5 & -7 \\ 2 & 5 & 0 & 2 & -3 & -1 \end{bmatrix}$$

After a lengthy calculation we find the row reduced echelon form R of M is given by

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Part (A): Is the system $Ax = b$ consistent?

Part (B): Find the row reduced echelon form (*aka*, Gauss–Jordan completely reduced row echelon canonical form) of A.

Part (C): Do the columns of A span $\mathbb{R}^4$?

Part (D): Do the columns of the matrix M span $\mathbb{R}^4$?

Problem 16. Suppose

$$M = \begin{bmatrix} 0 & 2 & 0 & 1 & -1 & -2 \\ 4 & -4 & 0 & -4 & -4 & 8 \\ 1 & 2 & 0 & 2 & 5 & -7 \\ 2 & 5 & 0 & 2 & -3 & -5 \end{bmatrix}$$

is the augmented matrix of a system of linear equations. After a lengthy calculation we find the row reduced echelon form R of M is given by

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Write the general solution of the linear system in vector form (*aka*, canonical parametric vector form).

Problem 17. Let $Ax = 0$ be a homogeneous system of 314 linear equations in 315 unknowns (variables). Explain carefully how you know that the system has a nontrivial solution. Your explanation (*aka*, proof) should probably contain the word *pivot*.

Problem 18. Let

$$v_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad v_4 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \quad .$$

Part (A): Write v_4 as a linear combination of $\{v_1, v_2, v_3\}$.

Part (B): Is the set $\{v_1, v_2, v_3, v_4\}$ linearly independent?

Part (C): Is the set $\{v_1, v_2, v_3\}$ linearly independent?

Problem 19. Let

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}.$$

Part (A): Find all solutions of the system $Bx = 2\,x$.

Part (B): Find all solutions of the system $Bx = 3\,x$.

Problem 20. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with $m \times n$ matrix A.

Part (A): Give a criterion, in terms of pivots of A, for T to be surjective (onto, epimorphic).

Part (B): Give a criterion, in terms of pivots of A, for T to be injective (one–to–one, monomorphic).

Part (C): Prove if $m = n$ the T is surjective if and only if T is injective. Is this statement true if $m \neq n$?

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3 Test 2

Problem 21. Let A be the 4×4 matrix

$$A = \begin{bmatrix} s & 1 & t & 1 \\ 0 & s & t & 0 \\ s & 1 & 1 & t \\ 0 & s & 1 & t \end{bmatrix}.$$

Part (A): Compute the determinant $\det(A)$ as an expression (simplified) in s and t.

Part (B): For what values of s and t is A invertible? Be sure to make precise use of "and" and "or" or in some other way state your conclusion precisely.

Problem 22. Let A be a 3×3 matrix. Suppose we augment A by adjoining the 3×3 identity matrix I to form the 3×6 matrix $[A, I]$. After doing a few elementary row operations on the matrix $[A, I]$ we end up with the matrix

$$\begin{bmatrix} 0 & 3 & 1 & 4 & 5 & 13 \\ 1 & 1 & 1 & 3 & 2 & 7 \\ 2 & 0 & 1 & 4 & 0 & 5 \end{bmatrix}.$$

Compute the inverse A^{-1} of A.

Problem 23. Let A be an $n \times n$ matrix. We say A is skew–symmetric if $A^T = -A$.

Part (A): Prove if n is odd and A is skew–symmetric then $\det(A) = 0$.

Part (B): Give an example of a 2×2 skew–symmetric matrix with determinant 1.

Problem 24. Let B be the 3×3 matrix

$$B = \begin{bmatrix} 0 & s & t \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix}.$$

Compute

$$I + B + 3\,B^2 - 5\,B^3 + 2\,B^{13} - B^{347}$$

where I is the 3×3 identity matrix.

Problem 25. The 3×3 matrix

$$A = \begin{bmatrix} 2 & 2 & 4 \\ 2 & -1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

has eigenvalues 8, -2, and -1. Find an eigenvector corresponding to the eigenvalue 8.

Problem 26. A certain 3×3 matrix A is known to have eigenvectors u_1, u_2, and u_3. If we form the 3×3 matrix $S = [u_1, u_2, u_3]$ we find

$$S = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad AS = \begin{bmatrix} 2 & 0 & 6 \\ 0 & 0 & 0 \\ -2 & 0 & 6 \end{bmatrix}.$$

Part (A): Find the eigenvalues of A.

Part (B): Find a matrix D such that $AS = SD$.

Problem 27. Let A be the 4×4 matrix

$$A = \begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

Find the eigenvalues of A. Without doing any calculating, explain how we know that there exists an invertible 4×4 matrix S and a diagonal 4×4 matrix D such that $S^{-1}AS = D$. There are 24 possible values for the diagonal matrix D. Give one of them.

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4 Contact Information

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