Linear Algebra – Mth 341

Archive – Spring 2000 Files

Feb 11, 2001

This archive contains the midterm test, an assignment and the final exam from Mth 241 Spring 2000. The original test instructions, headers and formatting have not been preserved.

Contents

1 Midterm Test

Problem 1. Consider a system of linear equations $A\vec{x} = \vec{b}$. After an immense number of elementary row operations we reduce the *augmented* matrix $[A, \vec{b}]$ to the row reduced echelon form,

$$\operatorname{rref}\Big([A,\vec{b}]\Big) = \begin{bmatrix} 0 & 1 & 0 & 0 & 3 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Call the variables in the system of linear equations $x_1, x_2, \cdots$

Part (A): How many variables are there?

Part (B): Which variables are free?

Part (C): Which variables are pivotal?

Part (D): What is the rank of the augmented matrix $[M, \vec{b}]$?

Part (E): Write the general solution of the system in vector parametric form.

Note: You may it convenient to the next 2 problems simultaneously.

Problem 2. Consider the vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 3 \\ a \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$. For what values of a is $\vec{v}$ in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3$?

Problem 3. Consider the vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 3 \\ a \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. For what values of a is $\vec{v}$ in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3$?

Problem 4. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and let B be a 2×2 matrix. If

$$AB = B + I$$

where I is the 2×2 identity matrix compute B.

Problem 5. A student playing around with some vectors $\vec{a}_k, k = 1, 2, 3, 4$, was amused to find

$$\vec{a}_1 + 2\vec{a}_2 + 3\vec{a}_3 + 4\vec{a}_4 = 4\vec{a}_1 + 3\vec{a}_2 + 2\vec{a}_3 + \vec{a}_4.$$

Are the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ linearly independent? Justify your answer.

Problem 6. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. First compute $(A^3 + 2 * A)^2$. Then compute

$$A\left((A^3 + 2A)^2 + 17I\right)^{1,423,846,679}$$

where I is the 2×2 identity matrix.

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2 Assignment

Do not show all your work. For example just say, "By row reducing I obtain $\cdots$," and then write the result you obtained. For other problems just sketch your work, emphasizing what you are doing rather than showing all the gruby details. But make sure you give clear and correct explanations.

Unclear, illegible *and late* assignments will loose credit.

Problem 7. Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 15 & -6 & 7 \\ 0 & 3 & -15 & -7 & 9 & 8 & 1 & 2 \\ 0 & -4 & 39 & 20 & -29 & 3 & -23 & 5 \\ 0 & -2 & 4 & 2 & 3 & -6 & 7 & -2 \end{bmatrix}.$$

Find the rank of A. What is the dimension of the null space $\text{nul}(A)$? What is the dimension of the row space $\text{row}(A)$? Which columns of A are pivotal?

Problem 8. Find all values of x such that the matrix

$$B = \begin{bmatrix} 1 & x & 1 \\ x & 1 & x \\ 2 & 3 & 5 \end{bmatrix}$$

is invertible.

Problem 9. Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 8 \\ -8 \\ 21 \\ -11 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ -4 \\ 3 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 6 \\ -7 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} -1 \\ 14 \\ 12 \\ -23 \end{bmatrix}.$$

Find the dimension of the subspace of $\mathbb{R}^4$ spanned by these vectors.

Problem 10. Let

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 4 & 4 & 1 \end{bmatrix}.$$

Let B be a 3×3 matrix such that

$$CB = C + B + I$$

where I is the 3×3 identity matrix. Find B.

Problem 11. Let

$$B = \begin{bmatrix} 289490605321453 & 142915498732717 \\ 142917092622049 & 70555200044678 \end{bmatrix}.$$

Compute the determinant $\det(B)$. If you do this calculation on a calculator explain why (or why not) that you may have a problem.

(Remark added to archive). The condition number of the matrix is about 1.87×10^{29}. Preliminary row reduction may be used to make the matrix more manageable (numerically). The determinant is 1.

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3 Final Exam

Problem 12. Compute the determinant of the matrix A given by

$$A = \begin{bmatrix} a & 1 & 2 \\ 1 & b & 3 \\ 2 & 3 & c \end{bmatrix}$$

and simplify. If $a = b = c = 0$ is the matrix A invertible?

Problem 13. Let A be a certain 3×3 matrix and let

$$S = \begin{bmatrix} 0 & 0 & 1/2 \\ 1 & 1 & 1 \\ 7/3 & 5/2 & 3/2 \end{bmatrix}.$$

Suppose

$$S^{-1}AS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Part (A): Find the eigenvalues and the corresponding eigenvectors for A.

Part (B): Find the inverse matrix S^{-1}.

Part (C): Compute the matrix A.

Problem 14. Let A be the 2×2 matrix

$$A = \begin{bmatrix} 5 & -3 \\ -4 & 3 \end{bmatrix}.$$

Part (A): Find the characteristic polynomial of A.

Part (b): Find the eigenvalues of A (exactly).

Problem 15. Consider the system of linear equations

$$\begin{aligned} 51794551x + ay &= 1 \\ 34459425x + by &= 1 \end{aligned}$$

where a and b are certain constants. Given that the coefficient matrix has determinant equal to 1, use Cramer's rule to find y.

Problem 16. Compute the dimension of the subspace W of $\mathbb{R}^4$ given by

$$W = \text{span}\left(\begin{bmatrix} 8 \\ -2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 11 \\ -1 \\ 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -9 \\ -4 \\ 3 \end{bmatrix}\right).$$

A.) 1

B.) 2

C.) 3

D.) 4

E.) None of the above.

☐ ←*Letter corresponding to your answer to problem 16.*

Problem 17. Let S be a set of 9 vectors in $\mathbb{R}^8$. Which of the following statements is (always) true?

A.) S is linearly independent

B.) S spans $\mathbb{R}^8$

C.) S is linearly dependent

D.) S does not span $\mathbb{R}^8$

E.) None of the above.

☐ ←*Letter corresponding to your answer to problem 17.*

Problem 18. Let S be a set of 7 vectors in $\mathbb{R}^8$. Which of the following statements is (always) true?

A.) S is linearly independent

B.) S spans $\mathbb{R}^8$

C.) S is linearly dependent

D.) S does not span $\mathbb{R}^8$

E.) None of the above.

☐ ←*Letter corresponding to your answer to problem 18.*

Problem 19. A certain homogeneous system of 7 linear equations in 9 unknowns has coefficient matrix A with rank 5. The solution space, $\text{nul}(A)$, has dimension

A.) 2

B.) 3

C.) 4

D.) 5

E.) None of the above.

☐ ←*Letter corresponding to your answer to problem 19.*

Problem 20. The matrix

$$B = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$$

is invertible if

A.) $a \neq 1$

B.) $a \neq -1$

C.) $a \neq 0$

D.) never

E.) None of the above.

☐ ←*Letter corresponding to your answer to problem 20.*

Problem 21. Write a *brief technical* essay of one or two paragraphs describing what you most enjoyed in linear algebra. Please use correct grammar and spelling. Make sure you include at least one theorem or technical result, and one or more definitions. An whatch dat spelung.

4 Contact Information

The contact information below is accurate as of Feb 11, 2001.

Bent E. Petersen
Department of Mathematics
Oregon State University
Corvallis, OR 97331-4605

phone numbers
office (541) 737-5163
home (541) 753-1829
fax (541) 737-0517

bent@alum.mit.edu
petersen@math.orst.edu
http://ucs.orst.edu/~peterseb
http://www.peak.org/~petersen
http://web.orst.edu/~peterseb