# Solution Sketches

## Problems 1 - 5 Mth 311 Fall 1997 Bent Petersen

Here are solution sketches for the first 5 problems. I have avoided giving you complete detailed solutions since you should provide your own.

**Problem 1.** Suppose the set $A$ has at least two elements. Choose $x \in A$ and $y \in A$ with $x \neq y$. Let $f$ be any function from the natural numbers to the set of sequences in $A$. Define a sequence $a$ by $a_n = y$ if $f(n)_n = x$ and $a_n = x$ otherwise. Then the sequence $a$ is not in the range of $f$.

**Problem 2**. Suppose $S$ is infinite. Choose $f(1) \in S$. Now inductively argue since $S$ is infinite there is a point in the set $S - \{f(1), f(2), ...f(n)\}$ .Choose one and define it to be $f(n+1)$.The rest is similar.

**Problem 3**. If the sequence $(a_n)$ does not converge to $a$ then there is $\epsilon > 0$ and a subsequence $(b_k)$ such that $|b_k - a| > \epsilon$ for each $k$. No subsequence of $(b_n)$ converges to $a$. For the second part each subsequence of the sequence $(-1)^n$ has a convergent subsequence.

**Problem 4**. Let $0 < c < 1$. Note if $c^n$ converges to $a$ then it also converges to $ca$ and so $a = 0$. Thus it suffices to prove that $c^n$ converges. That can be done by using monoticity, but invoking the completeness of the real numbers for this problem would be poor taste. The binomial expansion may be used to give a direct estimate. There is $p > 0$ such that $c = (1+p)^{-1}$. Now $(1+p)^n = 1 + np + n(n-1)n^2p^2 + ... \geq np$. Thus $c^n \leq \frac{1}{np}$.

**Problem 5**. If $0 < c < 1$ and $|a_{n+1} - a_n| \leq c^n$ then $|a_{n+k} - a_n| \leq c^n(1-c)^{-1}$. Now use the result of problem 4.