

Please turn in neat carefully written solutions to the problems. You should try to write good proofs. We are looking for technical details (check the “model” argument on the web page). You may discuss the problems with anyone for the purpose of obtaining ideas and clarification. You are expected however to produce and to write-up your own solutions.

Problem 9. Prove the *limit comparison* theorem: Let $a_n \geq 0$ and $b_n > 0$ for each n . Let

$$L = \liminf_{n \rightarrow \infty} \frac{a_n}{b_n} \quad \text{and} \quad U = \limsup_{n \rightarrow \infty} \frac{a_n}{b_n}.$$

1. If $U < \infty$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

Problem 10.

Part (A): Show if $a_n \geq 0$ for each n , $\sum_{n=1}^{\infty} a_n$ converges and $q \geq 1$ then $\sum_{n=1}^{\infty} a_n^q$ converges.

Part (B): Give an example of a divergent series $\sum_{n=1}^{\infty} b_n$ with $b_n \geq 0$ for each n such that $\sum_{n=1}^{\infty} b_n^q$ converges for each $q > 1$.

Problem 11. Prove that

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$$

converges.

Problem 12. Let $(a_n)_{n \geq 1}$ be a sequence of reals with $a = \liminf_{n \rightarrow \infty} a_n$. If $a > 1$ prove

$$\sum_{n=1}^{\infty} \frac{1}{n^{a_n}}$$

converges.

Problem 13. Use the root test to show that $\sum_{n=1}^{\infty} 2^{(-1)^n - n}$ converges. Does the ratio test have anything to say about this series?

Problem 14. Prove

$$\sum_{n=1}^{\infty} \frac{\log n}{n^2} \quad \text{converges and} \quad \sum_{n=1}^{\infty} \frac{1}{n^{(1+\frac{1}{n})}}$$

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