MTH 311 ASSIGNMENT 3 *Due: Oct 29, 1997*[*]

Please turn in neat carefully written solutions to the problems. You should try to write good proofs. We are looking for technical details (check the "model" argument on the web page). You may discuss the problems with anyone for the purpose of obtaining ideas and clarification. You are expected however to produce and to write–up your own solutions.

Problem 6. Let

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

for $n \geq 1$. Prove by induction that

$$s_n \leq \frac{61}{36} - \frac{1}{n}$$

for $n \geq 3$. Conclude that $s = \lim_{n\to\infty} s_n$ exists and $s \leq 61/36$.

Remark: One can show that $s = \pi^2/6 = 1.6449\cdots$. Since $61/36 = 1.694\cdots$ we are not too far off.

Problem 7. Define the sequence $(a_n)_{n\geq 1}$ inductively by $a_1 = 2$ and

$$a_{n+1} = \frac{a_n^2 + 2}{2a_n}$$

for $n \geq 1$. Prove that $a_n \geq \sqrt{2}$ for each $n \geq 1$. Next show the sequence is monotone decreasing and conclude that it converges. Finally prove that the limit is $\sqrt{2}$.

Hint: Show that $a^2 + 2 \geq 2\sqrt{2}a$ for any real number a, or more generally, $a^2 + b^2 \geq 2ab$ for any real numbers a and b.

Problem 8. Let

$$a_n = \frac{1 \times 3 \times 5 \cdots \times (2n-1)}{2 \times 4 \times 6 \cdots \times 2n} = \frac{(2n)!}{2^{2n}(n!)^2}.$$

Prove that $(a_n)_{n\geq 1}$ is a monotone decreasing sequence and that $a = \lim_{n\to\infty} a_n$ exists and $0 \leq a \leq 1/2$.

Let $b_n = \sqrt{n}a_n$. Prove that $(b_n)_{n\geq 1}$ is a monotone increasing sequence and $b = \lim_{n\to\infty} b_n$ exists. What does this imply about a?

Hint 1: In this problem the monoticity is most easily established by considering ratios.

Hint 2: Our study of $(1 + \frac{1}{n})^n$ leads to the estimate $1 + \frac{1}{n} \leq e^{1/n}$. Use this estimate to conclude

$$b_n \leq \frac{1}{2}e^{1/8}$$

and so $b = \lim_{n\to\infty} b_n$ exists and $b \leq \frac{1}{2}e^{1/8} = 0.566\cdots$. Actually one can show $b = 1/\sqrt{\pi} = 0.564\cdots$, so we are not too far off.

Bent E. Petersen	24 hour phone numbers
Department of Mathematics	office (541) 737-5163
Oregon State University	home (541) 754-2315
Corvallis, OR 97331-4605	fax (541) 737-0517

petersen@math.orst.edu
http://www.orst.edu/~peterseb

[*] File ref: 311F97A3.tex Fall 1997