Matrix and Power Series Methods
Sample Problems - Set 1

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Most, perhaps all, of the problems on the tests will be formulated as multiple-choice problems. However I have not bothered to do so for these sample problems.

Note that $\log(x)$ means the *natural* logarithm of x. If ever I need the logarithm base 10 of x I will denote it by $\log_{10}(x)$.

Problem 1. Determine if the improper integral

$$\int_0^\infty \frac{\mathrm{d}x}{\mathrm{e}^x + \mathrm{e}^{-x}}$$

converges or not. If it converges, determine its value.

Problem 2. Sum the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n + 2^n}{2^{2n}}.$$

Problem 3. Test for convergence

$$\sum_{n=0}^{\infty} \frac{4^n}{2^n + 3^n}.$$

Problem 4. Test for convergence

$$\sum_{n=0}^{\infty} \frac{3^n}{2^n + 4^n}.$$

Problem 5. Test for convergence

$$\sum_{n=0}^{\infty} \frac{2^{3n}(n!)^4}{((2n)!)^2}.$$

Problem 6. Use Stirling's formula to estimate (10!)!.

Problem 7. Test for convergence

$$\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^2}.$$

Problem 8. Test for convergence

$$\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right).$$

Problem 9. For what values of x does the series

$$\sum_{n=0}^{\infty} n!\, x^n$$

converge?

Problem 10. Test for convergence

$$\sum_{n=2}^{\infty} \frac{\log(n)}{n^2}.$$

Problem 11. Investigate the convergence or divergence of the improper integral

$$\int_1^{\infty} \frac{\mathrm{d}x}{x^{\frac{5}{4}}}.$$

Problem 12. Investigate the convergence or divergence of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} n^{-1/n}.$$

Problem 13. Investigate the convergence or divergence of the series

$$\sum_{n=0}^{\infty} \frac{2^{6n+3}(n!)^3}{3^n(3n)!}.$$

Problem 14. Investigate the convergence or divergence of the series

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{2n+3}{3n+2}\right)^{2n}.$$

Problem 15. Given

$$\mathrm{e}^{-1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$$

how many terms do we need to sum to approximate $\mathrm{e}^{-1/2}$ with an error no larger than 5×10^{-12}?

Problem 16. The Taylor-Maclaurin polynomial of degree 4 for a certain function g is given by

$$P(x) = 1 + 2x^2 - \frac{1}{6}x^3 - 3x^4.$$

Suppose we also know $\left| g^{(5)}(x) \right| \le 3.6$ for $|\,x\,| \le 1$. If we use $P(0.5)$ to estimate $g(0.5)$ compute the estimate and find a good upper bound for the error.

Here are some proposed answers. Note that the answers are not unique in many cases, that is, there is more than one way to do it. Note also that I have been known to make a mistake every now and then, so if we disagree check both your work and mine. In fact, part of your job is to check everything.

Answer 1. The integrand is dominated by e^{-x}. Hence the integral converges. The substitution $u = \mathrm{e}^x$ yields the value $\pi/4$.

Answer 2. $\frac{18}{7}$

Answer 3. Diverges by ratio test, limiting ratio is 4/3. Alternately you can use the simple comparison

$$\frac{4^n}{2^n + 3^n} \ge \frac{1}{2}\left(\frac{4}{3}\right)^n.$$

Answer 4. Converges by ratio test, limiting ratio is 3/4. Alternately you can use the simple comparison

$$\frac{3^n}{2^n + 4^n} \le \left(\frac{3}{4}\right)^n.$$

Answer 5. Converges by root test, limiting root is 1/2. Use Stirling's formula to compute the limiting root. You can also use the ratio test - the limiting ratio is 1/2.

Answer 6. According to Stirling

$$\log_{10}(n!) = \frac{1}{2}\log_{10}(2\pi n) + \left(n + \frac{1}{2}\right)\log_{10}(n) - n\log_{10}(e) + \frac{\delta}{12n}\log_{10}(e).$$

Replacing n by $10! = 3,628,800$ we obtain $\log_{10}(10!) = 2.222810395674425 \times 10^7 = 22,228,103.95674425$. It follows

$$(10!)! = 9.05199 \times 10^{22,228,103}.$$

Answer 7. Converges by integral test since substituting $u = \log(x)$ yields

$$\int_2^\infty \frac{\mathrm{d}x}{x(\log(x))^2} = \int_{\log 2}^\infty \frac{\mathrm{d}u}{u^2} = \frac{1}{\log 2}.$$

Answer 8. Since $0 < \sin(x) < x$ if $0 < x < \pi/2$ we have $\sin^2(1/n) < 1/n^2$ if $n > 1$. Thus we have convergence by comparison with the convergent p-series with $p = 2$.

Answer 9. If $x \neq 0$ then the absolute value of ratio of the $(n+1)^{\text{st}}$ term to the n^{th} term is $(n+1)\,|\,x\,|$ which tends to ∞ if $x \neq 0$. Thus we have convergence if and only if $x = 0$.

Answer 10. We know

$$\log(x) \leq \frac{x^p - 1}{p} \leq \frac{x^p}{p}$$

for any $x > 0$ and $p > 0$. In particular $\log(n) \leq 2n^{1/2}$. Thus

$$\frac{\log(n)}{n^2} \leq \frac{1}{n^{3/2}}.$$

Thus we have convergence by comparison with the p-series with $p = 3/2$.

Answer 11.

$$\int_1^b \frac{\mathrm{d}x}{x^{\frac{5}{4}}} = 4\left(1 - b^{-1/4}\right) \to 4$$

so the integral converges.

Answer 12.

$$\log\left(n^{-1/n}\right) = -\frac{\log n}{n} \to 0.$$

Thus $n^{-1/n} \to 1$ and so we have divergence by the 0-test.

Answer 13. Converges by the ratio test. The limiting ratio is $\frac{64}{81}$.

Answer 14. Converges absolutely by the root test. The limiting root is $\frac{4}{9}$.

Answer 15. The series is an alternating series with terms of decreasing magnitude with limit 0. Hence the error is bounded by the magnitude of the first term omitted. Thus we want to find n such that

$$\frac{1}{2^{n+1}(n+1)!} \leq 5 \times 10^{-12}. \quad \text{that is,} \quad 2^{n+1}(n+1)! \geq 2 \times 10^{11}.$$

Now

$$2^{11}11! = 8.2 \times 10^{10} \qquad 2^{12}12! = 2.0 \times 10^{12}.$$

Hence we need to take $n+1 = 12$, that is, $n = 11$. Since the terms are numerated from 0 we need 12 terms.

Answer 16. The estimate is $P(0.5) = 0.79167$ and the error in this estimate is bounded by

$$\frac{3.6}{5!}\left(\frac{1}{2}\right)^5 = 0.00094.$$

Note it is certainly also true that the error is bounded by 10^{300}, but in problems such as this one, you are expected to find a an error estimate by making a fairly close estimate of the Taylor remainder.

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