The CAUCHY initial value problem (IVP)

$$\begin{cases} \dfrac{dy}{dx} = x^2 - y^2 \\ y(1) = 2 \end{cases}$$

has a unique solution that can be written explicitly in terms of BESSEL functions. If we actually solve the problem and then compute $y(2)$ we find

$$y(2) = 1.75923881\ldots$$

Suppose however that we can not solve the problem explicitly in terms of functions known to us, then we can approximate $y(2)$ by using numeric methods, for example, EULER's method. The EULER method yields a polygonal approximation to the actual solution. Here is an example for IVP the problem above:

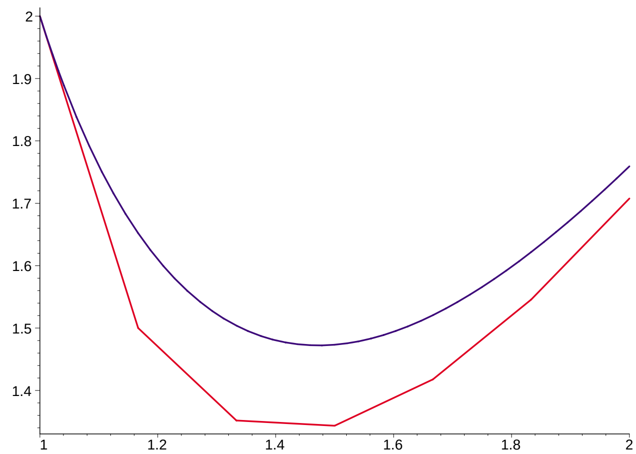

Figure 1: Six interval EULER polygonal approximation

The upper curve is the actual solution (generally unknown). The lower polygonal curve is the EULER approximation. The EULER approximation to $y(2)$ is

$$y_{\mathrm{a}}(2) = 1.70768026\ldots, \qquad \text{Error} = y(2) - y_{\mathrm{a}}(2) = 0.05155854\ldots$$

where $y_{\mathrm{a}}(2)$ is the approximate value given by EULER's method. The error is quite large but we can hope it will get smaller if we use a smaller step size, as long as we do not encounter excessive roundoff.

Problem 0113 – 1. Use EULER's method to compute $y_{\mathrm{a}}(2)$ for the cases of 5, 10 and 20 intervals. For each case report the value found and the corresponding error. What happens to the error as the step size is decreased (i.e., the number of intervals is increased). Try to give a rough quantitative answer.

Note, in solving this problem, you may use the true value of $y(2)$ given above to calculate the error. In practice, of course, the true value is not available and therefore the error has to be estimated. That estimation is part of the subject of numerical analysis, Mth 351, and will not concern us in Mth 256.