Weeks 1-3 Review
Fall 2006 Mth 256
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This review deals with recipes for solving first order ordinary differential equations as studied in the first three weeks of Mth 256, Fall 2006. The theoretical parts of the course are not reviewed here. The problems listed here do not represent every type of problem we discussed.

Answers indicated below were found mostly by Maple (for fun) and so may differ in some inessential way from the answers that you find. While I copy and paste the LaTeX version of the answers as provided by Maple, I can't help but clean up Maple's LaTeX expressions a bit. Undoubtably I introduced a few errors, so you should rely more on yourself than on the provided "answers."

Enjoy!

1 Picard Iteration

Problem 1. The initial value problem

$$\frac{dy}{dt} = t^2 + y^2, \quad y(0) = 1$$

can be solved in terms of certain Bessel functions. If we expand the solution in a Taylor series we obtain

$$y(t) = 1 + t + t^2 + \frac{4}{3}t^3 + \frac{7}{6}t^4 + \frac{6}{5}t^5 + \frac{37}{30}t^6 + \frac{404}{315}t^7 + \frac{369}{280}t^8 + \cdots$$

If you perform three Picard iterations (starting with $y_0(t) = 1$) what do you obtain? How does your result compare with the solution given above?

Answer:

$$1 + t + t^2 + \frac{4}{3}t^3 + \frac{5}{6}t^4 + \frac{8}{15}t^5 + \frac{29}{90}t^6 + \frac{47}{315}t^7 + \cdots$$

2 Linear First Order ODE

Problem 2. Solve the initial value problem

$$\frac{dy}{dt} + \tan(t)\, y = \tan(t), \quad y(0) = 3.$$

Answer:

$$y(t) = 1 + 2\cos(t)$$

Problem 3. Solve the initial value problem

$$\frac{dy}{dt} + \frac{y}{t} = \sin(t), \quad y(\pi) = 2.$$

Answer:

$$y(t) = \frac{\sin(t) - \cos(t)\,t + \pi}{t}$$

Problem 4.

$$\frac{dy}{dt} = \frac{y}{t-3} + t^2$$

Answer:

$$y(t) = (t-3)\left(\frac{t^2}{2} + 3t + 9\log(t-3) + C\right)$$

Problem 5.

$$\frac{dy}{dt} = \cot(t)\,y + \sin(t)$$

Answer:

$$y(t) = (t + C)\sin(t)$$

Problem 6.

$$\frac{dy}{dt} = \frac{y}{t} + \sin(\log(t))$$

Answer:

$$y(t) = -t\cos(\log(t)) + C\,t$$

Problem 7. A certain radioactive substance decays to 85 % of its original mass in 36 hours. Find the half–life.

Answer:

153.54 hours

3 Newton's Law of Cooling

Problem 8. A cup of hot coffee initially at temperature 134° F is brought into a room of temperature A (the ambient). The coffee begins to cool down and, of course the room warms up a bit. However, the heat capacity of the room is so large compared to the cup of coffee, that we may assume A remains constant. After 2 minutes the coffee is observed to have the temperature 100° K. Another minute later the coffee is observed to have the temperature 90° F. Deduce the temperature of the room.

Answer:

$$A = 65.64^\circ F.$$

Problem 9. A cup of coffee initially at 190° F is brought into a room at 65° F. After 2 minutes the temperature of the coffee is 145° F. Predict the temperature of the coffee an additional minute later.

Answer:

$$129^\circ \text{ F}$$

Problem 10. Consider an insulated box with internal temperature T. Assume that the ambient (external) temperature A is changing linearly (for a while at least), say

$$A = A_0 + A_1 t$$

where A_0 and A_1 are constants, and t is time. According to Netwon's law of cooling we have

$$\frac{dT}{dt} = -k\,(T - A)$$

where k is a constant depending on the insulation of the box. Find the temperature $T(t)$ in terms of t, A_0, A_1 and k. (Do not neglect the arbitrary constant.)

Answer:

$$T(t) = A_0 + A_1\left(t - \frac{1}{k}\right) + C\,\mathrm{e}^{-kt}$$

4 Mixing Problems

Problem 11. A brine solution consisting of 0.06 oz/gal salt dissolved in water flows into a large tank at the rate 3.0 gal/min. The solution inside the tank is kept well-mixed and flows out of the tank at the rate 2.0 gal/min. If the tank initially contains 50.0 gal of brine of concentration 0.03 oz/gal determine the amount of salt in the tank after t minutes. When will the concentration of salt in the tank reach 0.05 oz/gal? Assume the tank is so large that it does not overflow.

Answer:

22.1 min

Problem 12. A 200 L tank initially contains 100 L of brine of concentration 3 g/L salt (i.e., 3 grams salt per liter water). Brine of concentration 5 g/L salt runs into the tank at 8 L/min. The well–mixed solution is drawn off at the rate 6 L/min. Find the concentration of salt in the solution in the tank at the moment that the tank begins to overflow.

Answer:

4.875 g/L

Problem 13. A 100 gal tank initially contains 20 gal of brine of concentration 0.24 oz/gal salt. Brine of concentration 0.18 oz/gal flows into the tank at 3 gal/min and the well-mixed solution is drawn off at the rate of 1 gal/min. Find the amount of salt in the tank at the very moment that it begins to overflow.

Answer:

18.537 oz

5 Separable First Order ODE

Problem 14. For a body of mass m (say a person) falling near the surface of the earth we may assume the acceleration of gravity g is a constant. Also in many cases we may assume the drag is proportional to the square of the velocity. Thus the equation of motion is

$$m\frac{dv}{dt} = mg - kv^2.$$

Here v is the downward velocity. If we reparametrize this equation by the distance y fallen (try it) we obtain

$$mv\frac{dv}{dy} = mg - kv^2.$$

Assuming the body falls from rest find the velocity in terms of the distance fallen. From your solution determine the terminal, ultimate or limiting velocity (symbolically) if you fall a very long way. How far does a 60 kg person have to fall to reach 95 % of the terminal velocity if $k = 0.60$ kg/m? (Note k depends on the person's aspect. For feet or head first we may have roughly $k = 0.06$.)

Note you do not need the acceleration of gravity g to answer the questions above, but you do to compute the actual speed. Use $g = 9.80$ m/sec^2 to compute the 95 % of terminal velocity achieved by our intrepid aeronaut.

Answers:

$$v = \left(\frac{mg}{k}\right)^{1/2}\left(1 - \mathrm{e}^{-2ky/m}\right)^{1/2},$$

$$\left(\frac{mg}{k}\right)^{1/2},$$

and 116.4 m with a speed of 29.7 m/sec, or about 66.5 mph.

Problem 15. Solve the initial value problem

$$\frac{dy}{dx} = \frac{xy + y}{x}, \; y(1) = -2.$$

Answer:

$$y = -xe^{x-1}$$

Problem 16. Solve the initial value problem

$$\frac{dp}{dt} = \mathrm{e}^{2t}\,p(1-p), \quad p(0) = 0.5.$$

Find

$$\lim_{t\to\infty} p(t).$$

Answer:

$$p(t) = \frac{1}{1 + \mathrm{e}^{(1-\mathrm{e}^{2t})/2}}, \quad \text{The limit is 1.}$$

6 Bernoulli ODE

Problem 17. Solve the ordinary differential equation

$$y^3 \frac{dy}{dx} + \frac{y^4}{x} = \frac{\tan(x)}{x^4}.$$

Answer

$$x^4 y^4 + 4\log(\cos(x)) = C$$

Problem 18. Solve the ordinary differential equation

$$x\frac{dy}{dx} + 2y + \sqrt{y}\log(x) = 0.$$

Answer

$$\sqrt{y} + \frac{1}{2}\log(x) - \frac{1}{2} = \frac{C}{x}$$

7 Homogeneous First Order ODE and related substitutions

Problem 19. Use the substitution $y = x^2 w$ to solve the ordinary differential equation (not homogeneous)

$$\frac{dy}{dx} = \frac{2y^2 + x^3}{xy}.$$

Answer:

$$y^2 + 2x^3 + Cx^4 = 0$$

Problem 20. Solve the initial value problem

$$\frac{dy}{dx} = \frac{4x^2 + y^2}{2xy}, \quad y(2) = 2.$$

Answer:

$$y(x) = \left(4x^2 - 6x\right)^{1/2}$$

Problem 21. Solve the initial value problem

$$\frac{dy}{dx} = y(1+x^2) - (1+x^2), \quad y(0) = 2.$$

Answer:

$$y(x) = 1 + e^{x+\frac{x^3}{3}}.$$

(You can also solve this equation as a linear ode.)

8 Orthogonal Trajectories

Problem 22. Find the family of orthogonal trajectories to the one-parameter family of hyperbolas given by

$$2y^2 - x^2 = \alpha$$

.

Answer:

$$y = C/x^2$$

Problem 23. Find the family of orthogonal trajectories to the one-parameter family of cubics

$$y = \alpha x^3$$

Here α is the parameter, that is, an arbitrary constant.

Answer:

$$3y^2 + x^2 = \beta$$

Here β is a parameter, so we see we get a family of ellipses.

Problem 24. Consider the 1-parameter family of hyperbolas and ellipses given by

$$x^2 - \alpha y^2 = 1.$$

Here α is the parameter. Find the 1-parameter family of orthogonal trajectories.

Answer:

$$x^2 + y^2 - 2\log(x) = \beta$$

9 Integrating Factors and Exact ODE

Problem 25. For what values of p and q is $x^p y^q$ an intgrating factor for the ordinary differential equation

$$(6y^2 + 3y - 4xy)\,dx + (-3x^2 + 3x + 8xy)\,dy = 0?$$

Answer:

$$p = 2, q = 2$$

Note Maple finds the integrating factor $\mu = \frac{1}{xy(x-1-2y)}$. It should be possible to encourage Maple to also find $\mu = x^2y^2$, but I haven't had any luck.

Problem 26. Solve the *exact* ordinary differential equation

$$(2xy^3 - y^2 - 2) + (3x^2y^2 - 2xy + 3)\frac{dy}{dx} = 0.$$

Answer:

$$x^2y^3 - xy^2 - 2x + 3y = C$$

To get this answer in Maple you have to tell Maple you want an implicit solution. Otherwise Maple solves this cubic equation for y, which leads to quite a mess. So use a command like *dsolve(ode,implicit)*. If you have used *implicit* as a variable name, you'll have to escape it (using single quotes) to prevent evaluation (and confusion).

Problem 27. Find an integrating factor which depends only on y and then solve the differential equation

$$(2y + y^2 - 6xy) + (4x + 3xy - 6x^2)\frac{dy}{dx} = 0.$$

Answer:

$$y$$

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