Vector Calculus II – Mth 255

Archive – Winter 1998 Files

Feb 22, 2001

This archive contains the sample problems and tests from Mth 255 Winter 1998. The original test instructions, headers and formatting have not been preserved.

The sample problem sets were prepared using Scientific Notebook (just for fun) and included plots of curves and surfaces. For this archive I have just extracted the standard LaTeX 2_ε code from the Scientific Notebook LaTeX 2_ε sources and excluded special features and Maple drawing commands. Thus the original graphs are not included. If you have access to Maple you should try producing a few plots for yourself as you do the problems.

Contents

1 Sample Problem Set 1

Problem 1. Find a polynomial $f(x)$ of degree 5 such that its graph passes through the origin with slope 1 and through the point $(1, 0)$ with slope 0 and such that the curvature κ is 0 at the points $(0, 0)$ and $(1, 0)$. What does this problem have to do with railroad tracks?

Remark 1.1. A nonpolynomial solution may give a better distribution of the curvature in the sense of having a smaller maximum curvature. There's obviously more to learn before tackling the design of *real* tracks.

Problem 2. Plot the plane parametric curve $\vec{r} = \cos(2t)\vec{\mathrm{i}} + \sin(6t)\vec{\mathrm{j}}$. Compute the velocity, speed, unit tangent, acceleration, curvature, principal unit normal, and the tangential and normal components of the acceleration at $t = \pi/8$. Locate $\vec{r}(\pi/8)$ on the graph and check your calculations graphically (as well as you can).

Problem 3. Compute the maximum curvature of the graph of the cubic $y = x^3$.

Problem 4. Suppose a parametric curve has the property that at each point the position vector and the velocity vector are perpendicular. Show that the curve must lie on a sphere with center at the origin.

Problem 5. Show that a particle moves with constant speed if and only if its velocity and acceleration vectors are perpendicular at each point. Show in this case the curvature at each point is the magnitude of the acceleration vector divided by the square of the constant speed.

---

2 Sample Problem Set 2

Problem 6. Find the equation of the tangent plane to the surface $x^2 + xy + y^2 + z^2 = 37$ at the point $(3, 4, 0)$.

Problem 7. The paraboloid $z = 2x^2 + 3y^2$ intersects the cylinder $x^2 + y^2 = 1$ in a curve C. Find a tangent vector to the curve C at the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{5}{2})$.

Problem 8. Locate and classify the critical point of $f(x, y) = e^{xy}$.

Problem 9. Locate and classify the 2 critical points of $f(x, y) = \frac{x}{x^2+y^2+1}$.

Problem 10. Locate and classify the 4 critical points of $f(x, y) = (x + 2y + 3)xy$.

---

3 Sample Problem Set 3

Problem 11. Use the method of Lagrange multipliers to find the maximum value of $x + y$ on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are positive constants.

Problem 12. Let a, b, c be positive constants. Use the method of Lagrange multipliers to find the maximum value of $ax+by+cz$ on the sphere $x^2+y^2+z^2=1$. Now, in hindsight, give a geometric derivation of your answer.

Remark 3.1. You will find hindsight is not 20/20 unless you know how to compute the distance from a plane to the origin.

Problem 13. Let $p > 1$ and suppose $\frac{1}{p} + \frac{1}{q} = 1$. Find the maximum of

$$f(x_1, x_2, y_1, y_2) = x_1 y_1 + x_2 y_2$$

subject to the constraints

$$x_1^p + x_2^p = 1 \quad \text{and} \quad y_1^q + y_2^q = 1.$$

Note f is a function of four variables so you will have to think about how to extend the method of Lagrange multipliers to cover this case.

Problem 14. Compute the line integral

$$\int_C x ds$$

for each of the following plane curves C.

(A) C is the line segment from $(0,0)$ to $(3,1)$.

(B) C is the line segment from $(3,1)$ to $(0,3)$.

(C) C is the part of the parabola $x = y^2$ from $(1,1)$ to $(9,3)$.

Problem 15. Let C be the half cardioid $r = 1 - \cos(\theta)$ $0 \le \theta \le \pi$, that is

$$\begin{aligned} x &= \cos(\theta) - \cos^2(\theta) \\ y &= \sin(\theta) - \sin(\theta)\cos(\theta) \end{aligned}$$

Compute the line integral

$$\int_C y dx - x dy \tag{1}$$

(After a lengthy calculation you should perhaps get $-3\pi/2$).

Compute also the line integral

$$\int_C \frac{y dx - x dy}{\sqrt{x^2 + y^2}}$$

(After a length calculation or a brief insight you should get $-\pi$).

Problem 16. Compute the line integral

$$\int_C x^2 y ds$$

where C is the quarter astroid

$$\begin{aligned} x &= \cos^3(\theta) \\ y &= \sin^3(\theta) \end{aligned}$$

$0 \leq \theta \leq \pi/2$.

Remark 3.2. This is a lengthy calculation - don't give up too soon. The answer is probably $\frac{16}{143}$.

Problem 17. Part of the folium of Descartes

$$x^3 + y^3 = 3xy$$

may be parametrized as

$$\begin{aligned} x &= \frac{3t}{1+t^3} \\ y &= \frac{3t^2}{1+t^3} \end{aligned}$$

Compute the line integral

$$\int_C x dy$$

where C is the part of the folium traversed as t goes from 0 to 2 in the above parametrization. (The answer is probably $\frac{16}{9}$.) How about as t goes from 0 to ∞? What part of the curve is traced out in this case? (The answer is probably $\frac{3}{2}$).

4 Sample Problem Set 4

Problem 18. Let

$$\vec{F} = (2xy^4z^2 + z)\vec{i} + 4x^2y^3z^2\vec{j} + 2x\vec{k}$$

Show that $z\vec{F}$ is conservative, but $\vec{F}$ is not conservative. Find a potential f for $z\vec{F}$, that is, a function f such that

$$\vec{\nabla} f = z\vec{F}$$

Problem 19. Let Ω be the region between $x = 0$ and $x = \pi$ bounded by the x-axis and the graph of $y = \sin(x)$. Let C the boundary of Ω traversed once counter-clockwise. Use Green's theorem to compute the line integral

$$\oint_C (xy + x^3)dx + (6y + y^5)dy$$

Problem 20. If $\vec{F} = x^2y\vec{\imath} + xy^2\vec{\jmath} + xyz\vec{k}$ compute curl($\vec{F}$). Can you quickly compute div(curl($\vec{F}$))?

Problem 21. Compute div(grad(exp(x) sin(y))).

Problem 22. Consider the parametric surface

$$\vec{r} = u\cos(v)\vec{\imath} + u\sin(v)\vec{\jmath} + u\vec{k}.$$

Find a normal at the point $(1, 0, 1)$. Find an equation of the tangent plane at this point. Find the surface area of the portion of the surface determined by $0 \leq u \leq 2$ and $0 \leq v \leq \pi$.

Problem 23. The surface of a torus may be parametrized as

$$\vec{r}(\theta, \alpha) = (a + b\cos\alpha)\cos\theta\vec{\imath} + (a + b\cos\alpha)\sin\theta\vec{\jmath} + b\sin\alpha\vec{k}$$

where a, b are constants with $a > b$, $0 \leq \alpha \leq 2\pi$ and $0 \leq \theta \leq 2\pi$. Compute the surface area element $dS = |\vec{r}_\theta \times \vec{r}_\alpha|\, d\theta d\alpha$ and then compute the surface area of the torus.

We may view the tous as swept out by a moving circle of radius b. Then an ancient theorem of Pappus says the area swept out is equal to the arc length of the circle times the distance the centroid of the circle moves. Use this result to check your answer above.

Show the outward unit normal to the torus is

$$\vec{n} = \cos\theta\cos\alpha\vec{\imath} + \sin\theta\cos\alpha\vec{\jmath} + \sin\alpha\vec{k}.$$

Compute the flux

$$\iint_S \vec{F} \cdot \vec{n}\, dS$$

of the vector field $\vec{F} = x\vec{\imath} + y\vec{\jmath}$ across the surface of the torus.

Here is the Scientific Notebook help entry on surface area. It is a nice review.

Areas of Parametric Surfaces

- Just as a space curve is described by a vector function $\mathbf{r}(t)$ of one parameter, a surface is described by a vector function of two parameters. Let S denote a surface given by the vector equation

  $$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D.$$

  Assume that the partial derivatives of x, y, and z with respect to u and v are all continuous.

- If you fix one of the parameters and let the other vary, you get a space curve that lies in the surface. The vectors
$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial y}{\partial u}\mathbf{j} \quad \text{and} \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial y}{\partial v}\mathbf{j}$$
are tangents to these "coordinate curves", so both lie in the tangent plane to the surface. If you evaluate $\mathbf{r}_u$ and $\mathbf{r}_v$ at some point on the surface and calculate the cross product $\mathbf{r}_u \times \mathbf{r}_v$ you will get a vector **normal** (i.e. perpendicular) to the surface at that point.

- If you assume that each point of the surface corresponds to a distinct point of D, the formula for the **area of the surface** is
$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$
In the special case in which D is a domain in the x-y plane and the surface is given explicitly by the formula $z = f(x, y)$, the parametrized description of the surface is
$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\,\mathbf{k} \qquad (x, y) \in D.$$
and the formula for surface area in this case is
$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy$$

5 Test 1

Problem 24. For the parametric curve
$$\vec{r}(t) = t^3\,\vec{\mathrm{i}} - 2t^2\,\vec{\mathrm{j}} + (t^2 + 1)\,\vec{\mathrm{k}}$$
compute each of the following quantities. In this problem, and the rest, show (and explain) your work. Be sure to consider ways to reduce the amount of calculation in this problem. Otherwise it will be very long.

Part (A): the velocity at "time" $t = 2$

Part (B): the speed at "time" $t = 2$

Part (C): the unit tangent at "time" $t = 2$

Part (D): the acceleration at "time" $t = 2$

Part (E): the curvature at "time" $t = 2$

Part (F): the tangential component of acceleration at "time" $t = 2$

Part (G): the normal component of acceleration at "time" $t = 2$

Part (H): the principal unit normal at "time" $t = 2$

Part (I): the equation of the osculating plane at "time" $t = 2$

Part (J): the radius of curvature at "time" $t = 2$

Problem 25. Find all the critical points of the function

$$f(x, y,) = x^3 + y^3 - 3\,x - 12\,y + 17.$$

Use the second derivative test for functions of two variables to classify the critical points you have found.

Problem 26. Use the method of Lagrange multipliers to find the maximum of x^2y on the circle $x^2 + y^2 = 12$.

Problem 27. Compute the following line integrals:

Part (A): $\oint_C x\,dy$ where C is the circle of radius 1 with center at the origin and traversed once counterclockwise.

Part (B): $\oint_C x^2y^2\,ds$ where C is the circle of radius 1 with center at the origin and traversed once counterclockwise.

Part (C): $\int_C xyz\,dy$ where C is the line segment from the point $(1, 1, 1)$ to the point $(2, 3, 4)$.

Part (D): $\oint_C \vec{\nabla} f \cdot d\vec{r}$ where f is a continuously differentiable function of three variables and C is a closed curve in three–space.

---

6 Test 2

Problem 28. Let Ω be the region in the first quadrant bounded by the parabolas $y = x^2$ and $y = 3x^2$, and by the hyperbolas $xy = 2$ and $xy = 4$. Make the change of variables

$$\begin{aligned} x &= u^{-1/3}\,v^{1/3} \\ y &= u^{1/3}\,v^{2/3} \end{aligned}$$

to evaluate the integral

$$\iint_\Omega y^2x^{-1}\,dxdy.$$

Note that $u = yx^{-2}$ and $v = xy$.

Problem 29. **Part (A):** If possible, find a potential f for the vector field

$$\vec{F} = (x^2 + y^2 + 2zx)\vec{\mathrm{i}} + (y^2 + z^2 + 2xy)\vec{\mathrm{j}} + (z^2 + x^2 + 2yz)\vec{\mathrm{k}}.$$

Part (B): Compute the integral $\int_C \vec{F} \cdot d\vec{r}$ where C is the line segment from $(0, 0, 0)$ to $(1, 2, 1)$.

Problem 30. Use Green's theorem to compute the line integral

$$\oint_C (y + x^5)dx + (6y + y^5 + y^7)dy$$

where C is the circle $(x - 1)^2 + y^2 = 1$ traversed counterclockwise once.

Problem 31. Compute the surface integral

$$\iint_S z^{-3} \left(x^2 + y^2 + z^{-4}\right)^{-1/2} \, dS$$

where S is the portion of the hyperboloid $xyz = 1$ which lies above the rectangle $1 \le x \le 2$, $1 \le y \le 3$.

---

7 Dad's Weekend Quiz

This quiz was given to keep visiting Dad's busy. Hopefully it brought back fond memories $\cdots$

Problem 32. Write an essay describing your reaction to the O.S.U. campus and what you enjoy most about Dads' Weekend.

Problem 33. Let a, b, c be positive constants. Use geometric reasoning or the method of Lagrange multipliers to find the maximum value of $ax + by + cz$ on the sphere $x^2 + y^2 + z^2 = 1$.

---

8 Final Exam

We will use $\log x$ to denote the natural logarithm of x (a.k.a. $\ln x$). The logarithm base 10 of x, if needed, will be denoted by $\log_{10} x$.

$$\begin{aligned} \operatorname{grad} f &= \vec{\nabla} f \\ \operatorname{curl} \vec{F} &= \vec{\nabla} \times \vec{F} \\ \operatorname{div} \vec{F} &= \vec{\nabla} \cdot \vec{F} \end{aligned}$$

Problem 34. *(70 points).* Let

$$\vec{F}(x,y,z) = \mathrm{e}^{\tan z} y\,\vec{\mathrm{i}} + \mathrm{e}^{z\tan z} x\,\vec{\mathrm{j}} + \left(4 - x^2 - y^2\right) z\,\vec{\mathrm{k}}.$$

Let Ω be the region which lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 4$. Let S be the boundary of Ω and let $\vec{n}$ be the outward unit normal of S. Use the divergence theorem to compute the total flux

$$\iint_S \vec{F} \cdot \vec{n}\, dS$$

of $\vec{F}$ outward through S.

Problem 35. *(80 points).* Let S be the surface consisting of the part of the paraboloid $x^2+y^2+z=9$, $z \geq 0$, which lies inside the cylinder $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$. Suppose $\vec{n}$ is the upward unit normal of S. Use Stokes' theorem to compute

$$\iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS$$

where $\vec{F} = y\,\vec{\mathrm{i}} + x\,\vec{\mathrm{j}} + xy\,\vec{\mathrm{k}}$. Hint: You might want to parametrize the boundary C of S by $x = 3\cos t$, $y = 2\sin t$ and $z =$(you figure it out).

Problem 36. *(70 points).* Calculate the surface integral

$$\iint_S z^2\, dS$$

where S is that part of the paraboloid $z = 9 - x^2 - y^2$ which lies above the plane $z = 0$.

Problem 37. *(70 points).* If C is the cardioid given in polar coordinates by $r = 1 + \cos\theta$ use Green's theorem to evaluate the line integral

$$\int_C xy\, dx + x^2\, dy$$

where we traverse the cardioid once in the counter–clockwise direction.

Problem 38. *(70 points).* Let R be the region in the plane which lies between the parabolas $y = 2x^2$ and $y = 4x^2$ and also between the parabolas $x = y^2$ and $x = 3y^2$. Use the change of variables $x = u^{2/3}v^{1/3}$ and $y = u^{1/3}v^{2/3}$ to evaluate the integral

$$\iint_R xy\, dx dy.$$

Problem 39. *(80 points).* Use the method of Lagrange multipliers to find the maximum and the minimum of z^2 subject to the constraints $x + y + z = 1$ and $x^2 + y^2 = 8$.

9 Contact Information

The contact information below is accurate as of Feb 22, 2001.

Bent E. Petersen	phone numbers
Department of Mathematics	office (541) 737-5163
Oregon State University	home (541) 753-1829
Corvallis, OR 97331-4605	fax (541) 737-0517

`bent@alum.mit.edu`
`petersen@math.orst.edu`
`http://ucs.orst.edu/~peterseb`
`http://www.peak.org/~petersen`
`http://web.orst.edu/~peterseb`