Of Rabbits, Graphs and Eigenvalues

Mth 254 – Bent E. Petersen

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In Mth 254 we spend 2 or 3 lectures on the eigenvalues and eigenvectors of a matrix, and on the notion of diagonizability of a matrix. There are many important applications in geometry, differential equations, engineering, physics and chemistry of the notion of eigenvalues of a matrix, but of course we have no time to discuss them. In addition these applications require considerable background beyond calculus and linear algebra.

Here are two simple, but non–trivial, applications of these notions. These applications are quite interesting but require no background in any other discipline or area of mathematics. They are rather offbeat, but perhaps not unimportant.

Rabbits

Leonardo of Pisa, also known as Fibonacci, lived from about 1170 to 1250. He was educated in Africa where much of the legacy of the Greeks had been preserved and further developed. In 1202 he wrote the *Liber Abaci* which was instrumental in bringing Arabic, Arabic–Greek, and Hindu arithmetic, geometry, and algebra into Europe, and in popularizing the concept of zero. He also showed that the Greek theory of irrationals is insufficient, that a certain cubic (with integer coefficients) has roots that can not be constructed in the Greek sense. Without a doubt Fibonacci was very important in the development of European mathematics.

Fibonacci also promulgated a model for the propogation of rabbits. I do not know if he was serious, or making a joke, but his model is nowadays the main result by which he is commemorated. It leads to a sequence of numbers in which each term is the sum of the preceding two terms. Such sequences, and related ones, have remarkable properties.

Consider now the sequence of Fibonacci numbers,

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1},\, n \geq 2.$$

The first few terms in the sequence F_n are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$$

This sequence is related to matrix multiplication in the following way. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

What is obvious about the matrix A? Well, it is symmetric and so is diagonizable. We can easily compute the eigenvalues of A (**Exercise.** Do it.):

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

Moreover since $\lambda_k^2 = 1 + \lambda_k$ we see that if

$$S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

then

$$A = SDS^{-1}$$

where

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

It follows that (**Exercise.** Do it.)

$$A^n = SD^nS^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & \lambda_1^n - \lambda_2^n \\ \lambda_1^n - \lambda_2^n & \lambda_1^{n-1} - \lambda_2^{n-1} \end{pmatrix}$$

It follows that $F_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}}$. This is just the well–known *Binet formula*

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

The number $\frac{1+\sqrt{5}}{2}$ which occurs in Binet's formula is the celebrated *golden ratio* of the Greeks which shows up in diverse subjects. For example, the ratio of a person's height to the height of the navel is supposed to be about equal to the golden ratio. (**Exercise.** Get a measuring tape and check this claim.)

Graphs

By a graph we mean an ordered set of n points, called the *vertices*, some of which are joined by abstract line segments, called the *edges*. A graph can be completely described by its $n \times n$ *adjacency matrix* $A = (a_{ij})$. We define $a_{ij} = 1$ if the i^{th} and j^{th} vertices are joined by and edge: otherwise $a_{ij} = 0$.

If we imagine ourselves travelling from one vertex to another along the edges, possibly moving back and forth on the same edge several times, then we call the total number of edges traversed the *length* of our path. We define $a(m,i,j)$ to be the number of paths of length m from the i^{th} to the j^{th} vertex. Thus $a(1,i,j) = a_{ij}$. It is an excellent **Exercise.**in matrix multiplication (and mathematical induction) to show that

$$A^m = (a(m,i,j))_{1\le i,j\le n}\,.$$

The number of paths of a given length turns out to be important in some areas of physics (that to me seem even more abstract than mathematics).

Now the adjacency matrix A is obviously symmetric and so diagonizable. That is we can find a matrix S suct that

$$S^{-1}AS = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$$

where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A. It follows then that

$$A^m = S\,\mathrm{diag}(\lambda_1^m, \cdots, \lambda_n^m)\,S^{-1}.$$

This expression gives us an explicit formula for the $a(m,i,j)$ – a formula which is analogous to the Binet formula for the Fibonacci numbers.

As an example let A be the adjacency matrix of an ordinary triangle (also known as a *2–simplex*). Then

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then eigenvalues of A are $2, -1, -1$ (**Exercise.** Check!). For S we can take

$$S = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Now S^{-1} is readily computed (**Exercise.** Do it.) and the formula for A^m then yields:

$$a(m,i,j) = \begin{cases} \frac{2^m + 2(-1)^m}{3} & \text{if } i = j \\ \frac{2^m - (-1)^m}{3} & \text{if } i \neq j \end{cases}.$$

For a tetrahedron (a.k.a a *3–simplex*) we obtain in like manner (**Exercise.** Do it.)

$$a(m,i,j) = \begin{cases} \frac{3^m + 3(-1)^m}{4} & \text{if } i = j \\ \frac{3^m - (-1)^m}{4} & \text{if } i \neq j \end{cases}.$$

Perhaps you can come up with (and prove) a general formula for the *p-simplex.* It's the high degree of symmetry in these examples that makes the calculations reasonably easy.

Consider now a tetrahedron with one edge *broken.* For example, suppose we remove the edge joining the first and fourth vertex. Then the adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

For this matrix the eigenvalues are

$$-1, 0, \frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}.$$

We now have

$$S = \begin{pmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & \alpha & \beta \\ 1 & 0 & \alpha & \beta \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

where

$$\alpha = \frac{1+\sqrt{17}}{4}, \quad \beta = \frac{1-\sqrt{17}}{4}.$$

It's not too difficult to compute S^{-1} (especially if you use Maple) and therefore, as before, to obtain explicit formulae for the components of A^m. The formulae are quite complicated however. (**Exercise.** Try to obtain the formulae.)

If we break two edges of the tetrahedron we obtain a triangle with a tail. The adjacency matrix now is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

In this case the formulae are very complicated. Note we can of course compute A^m for any m, but that is not very exciting. For example for our tailed–riangle we have

$$A^{20} = \begin{pmatrix} 1466055 & 1714550 & 1466054 & 790748 \\ 1714550 & 2008307 & 1714550 & 923802 \\ 1466054 & 1714550 & 1466055 & 790748 \\ 790748 & 923802 & 790748 & 426811 \end{pmatrix}.$$

Unless we have a burning interest in paths of length 20 this result is not of much use (unlike the formulae above). Even so, it beats trying to count the paths directly.

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