Vector Calculus I – Mth 254

Archive – Fall 1998 Files

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This document contains 28 sample problems, the midterm test and the final exam from Mth 254 Fall 1998. The original test formatting has not been preserved.

Contents

1 Sample problems - set 1

The problems presented here are for study and practice. They are not represented as being indicative of the problems which will be on our test.

Problem 1. Use polar coordinates to evaluate the integral

$$\iint_{\Omega} \frac{1}{\sqrt{9+x^2+y^2}}\,dxdy$$

where Ω is the disc in the plane with radius 4 and with center at the origin.

Problem 2. Let Ω be a region in the x, y-plane. Find the surface area of the portion of the cone $z^2 = x^2 + y^2$ which lies above the region Ω.

Problem 3. Find the surface area of the portion of the surface $z = x^2 - y^2$ which lies above the disc bounded by the circle $x^2 + y^2 = 6$ in the x, y-plane.

Problem 4. Compute

$$\iint\limits_{\Omega} \sqrt{x^2 + y^2}\, dxdy$$

where Ω is the region which lies inside the cardioid described in polar coordinates by

$$r = a(1 + \cos\theta), \quad 0 \le \theta \le 2\pi, \quad (a > 0 \text{ is a constant}).$$

Problem 5. Let T be the tangent plane to the surface

$$z = 2x^2 + 3y^2 + 50$$

at the point (a, b, c) where $c = 2a^2 + 3b^2 + 50$. Find the distance from T to the origin.

Problem 6. Change the order of integration in the iterated integrals

$$\int_0^1 \int_{x^2}^{4x^2} f(x, y)\, dy\, dx + \int_1^2 \int_{x^2}^{4} f(x, y)\, dy\, dx.$$

Problem 7. Find the equation of the plane tangent to the graph of

$$z = x^2 - y^2 + 2xy + \mathrm{e}^{-x} \cos y - y$$

at the point $(0, 0, 1)$.

Problem 8. Let

$$\begin{aligned} z &= x\mathrm{e}^y + 4\sin\left(\frac{x}{y}\right) \\ x &= 2uv \\ y &= u^2 + v^2 + 3 \end{aligned}$$

Use the chain rule to compute $\dfrac{\partial z}{\partial u}$ at $(u_0.v_0) = (0, 1)$.

Problem 9. Compute the integral

$$\iint_{\Omega} \left(1 - \sqrt{x^2 + y^2}\right)\, dxdy$$

where Ω is the semidisc consisting of points (x, y) with $x^2 + y^2 \le 1$ and $y \ge 0$.

Problem 10. Change to polar coordinates in each of the following iterated integrals

1. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\, dy\, dx$
2. $\displaystyle\int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x,y)\, dy\, dx$
3. $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\, dx\, dy$
4. $\displaystyle\int_0^{\sqrt{2}/2} \int_x^{\sqrt{1-x^2}} f(x,y)\, dy\, dx.$

Problem 11. Compute the integral

$$\int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2 + y^2 + 1}}\, dy\, dx.$$

Problem 12. Compute the integral

$$\int_0^4 \int_{x/2}^2 \mathrm{e}^{y^2}\, dy\, dx$$

by first changing the order of integration.

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2 Midterm test

Problem 13. Find the slope of the tangent line to the graph of the relation

$$2x^4 + 3x^2y^2 - 2y^3 + 3x - 6y = 0$$

at the point $(1, 1)$.

Problem 14. Find all the points $(1, b, c)$ on the graph of

$$z = 2x^2 + 3y^2 + 50$$

such that the tangent plane at $(1, b, c)$ passes through the origin.

Problem 15. Change the order of integration in the iterated integral

$$\int_0^2 \int_0^{2x^2} f(x, y)\ \mathrm{d}y\,\mathrm{d}x.$$

Problem 16. Use polar coordinates to evaluate the double integral

$$\iint_\Omega \frac{y^2}{\sqrt{9 + x^2 + y^2}}\ \mathrm{d}x\,\mathrm{d}y$$

where Ω is the disc with center at the origin and radius 4.

Problem 17. Find the surface area of that portion of the hyperbolic paraboloid

$$z = 12 + xy$$

that lies above the disc with center at the origin and radius $\sqrt{24}$.

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3 Sample problems - set 2

The problems presented here are for study and practice. They are not represented as being indicative of the problems which will be on our test. Note problem number 5 is not suitable for a test. It is included for fun and to hone your matrix skills.

Problem 18. Let the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ be the successive columns of the matrix

$$A = \begin{bmatrix} 3 & 4 & 4 \\ 5 & 3 & 5 \\ 3 & -2 & 2 \end{bmatrix}.$$

Write the vector

$$\vec{v} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$$

as a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$ and compute the coefficients.

Problem 19. The vector

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

is known to be an eigenvector of the matrix

$$A = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}.$$

Find the corresponding eigenvalue.

Problem 20. The scalar 2 is known to be an eigenvalue of the matrix

$$A = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}.$$

Find a corresponding eigenvector.

Problem 21. Let

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & -2 \\ 8 & -11 & 13 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and } \vec{b} = \begin{bmatrix} 2 \\ t \\ -1 \end{bmatrix}.$$

Use row reduction to find all values of t for which the matrix equation $A\vec{x} = \vec{b}$ has at least one solution.

Problem 22. (Original problem 5) Let A be the 2×2 matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Show that

$$A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix},$$

where F_n is the n^{th} Fibonacci number, $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $\cdots$, $F_{20} = 6765$, and in general $F_{n+1} = F_n + F_{n-1}$. Show that the eigenvalues of A are

$$\frac{1+\sqrt{5}}{2}, \quad \frac{1-\sqrt{5}}{2}.$$

Find a matrix S such that $S^{-1}AS = D$ where

$$D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}.$$

Conclude $A^n = SD^nS^{-1}$. Find an explicit expression for D^n. Use it to compute A^n and then conclude

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

This formula is known as the Binet formula.

Problem 23. Suppose you have a certain 3×3 invertible matrix A and you wish to compute the inverse. You augment A by the identity matrix I, spend a pleasant few moments row-reducing, and obtain

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 3 & 1 \\ 3 & 4 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{bmatrix}.$$

Finish the job and find A^{-1}. How would you find A?

Problem 24. Find the eigenvalues of the matrix

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Does B have any real eigenvectors?

Problem 25. Find the eigenvalues of the matrix

$$C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Is there a basis of $\mathbb{R}^2$ consisting of eigenvectors of C?

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4 Sample problems - set 3

The problems presented here are for study and practice. They are not represented as being indicative of the problems which will be on our test.

Problem 26. Let $a > 0$. Use spherical coordinates to compute the triple integral

$$\iiint\limits_{\Omega} \left(x^2 + y^2 + z^2\right)^n \;\, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where

$$\Omega = \left\{\, (x, y, z) \;\mid\; x^2 + y^2 + z^2 \le a^2,\;\; z \ge 0 \,\right\}.$$

Problem 27. Let $a > 0$. Use spherical coordinates to compute the triple integral

$$\iiint_{\Omega} \left(x^2+y^2\right)^2 \;\; \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where

$$\Omega = \left\{\, (x,y,z) \;\mid\; x^2+y^2+z^2 \le a^2,\;\; z \ge 0 \,\right\}.$$

Problem 28. Let $a > 0$. Use spherical coordinates to compute the triple integral

$$\iiint_{\Omega} \left(x^2+y^2\right) \;\; \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where

$$\Omega = \left\{\, (x,y,z) \;\mid\; x^2+y^2+z^2 \le a^2,\;\; z \ge 0 \,\right\}.$$

Problem 29. Let $a > 0$. Use spherical coordinates to compute the triple integral

$$\iiint_{\Omega} \left(x^2+y^2\right)^3 \;\; \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where

$$\Omega = \left\{\, (x,y,z) \;\mid\; x^2+y^2+z^2 \le a^2,\;\; z \ge 0 \,\right\}.$$

Problem 30. Let $a > 0$. Use cylindrical coordinates to compute the triple integral

$$\iiint_{\Omega} \mathrm{e}^{a^2-x^2-y^2} \;\; \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where

$$\Omega = \left\{\, (x,y,z) \;\mid\; x^2+y^2+z^2 \le a^2,\;\; z \ge 0 \,\right\}.$$

Problem 31. Use cylindrical coordinates to compute the triple integral

$$\iiint_{\Omega} \left(x^2+y^2\right)^n \;\; \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where Ω is the bounded region between the paraboloids $z = 2 - x^2 - y^2$ and $z = x^2 + y^2 - 2$.

Problem 32. Use cylindrical coordinates to compute the triple integral

$$\iiint_\Omega \left(x^4 - y^4\right)^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where Ω is the region between the planes $z = -1$ and $z = 1$, and inside the cylinder $x^2 + y^2 = 1$.

Problem 33. Use cylindrical coordinates to compute the triple integral

$$\iiint_\Omega \left(x^4 - y^4\right)^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where Ω is the bounded region between the paraboloids $z = 2 - x^2 - y^2$ and $z = x^2 + y^2 - 2$.

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5 Final exam

Problem 34. For each of the following 2×2 matrices determine for what value(s) of the parameter h that the matrix is invertible.

$$A = \begin{bmatrix} h+1 & h-1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} h+1 & h+1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & h-1 \\ h+2 & 4 \end{bmatrix}$$

Problem 35. The scalar 8 is known to be an eigenvalue for the matrix

$$A = \begin{bmatrix} 5 & 3 & 3 \\ 3 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}.$$

Find a corresponding eigenvector.

Problem 36. Le $a > 0$ and $b > 0$. Find the centroid of the quarter ellipse consisting of the set of points (x, y) such that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1, \quad x \ge 0, \quad y \ge 0.$$

Problem 37. Let $a > 0$ be such that $3 \log a = \log \log 3$. Use spherical coordinates to compute the triple integral

$$\iiint_\Omega \mathrm{e}^{(x^2+y^2+z^2)^{3/2}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where Ω is the ball consisting of points (x, y, z) such that $x^2 + y^2 + z^2 \le a^2$.

Problem 38. Suppose you have a certain 3×3 invertible matrix A and you wish to compute the inverse. You augment A by the identity matrix I, spend a few pleasant moments randomly row-reducing, and obtain

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & -4 \\ 1 & 1 & 3 & 3 & 1 & -4 \\ 4 & 2 & 1 & 5 & 6 & -13 \end{bmatrix}.$$

Finish the job and find A^{-1}.

Problem 39. **Part (A):** Find the equation of the tangent plane to the quintic surface S

$$z = 3x^3y^2 - 2x^2y^3 + 3x^2y + 3xy^2 + 2x - 3y$$

at the point $(1, 1, 6)$. **Part (B):** Now find the equation, in parametric form, of the line through the point $(1, 1, 6)$ orthogonal to A.

Problem 40. Use cylindrical coordinates to compute the triple integral

$$\iiint_{\Omega} (x^2 - y^2)\, z \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

where Ω is the region in space bounded above by part of the paraboloid $z = 4 - x^2 - y^2$ and bounded below by the portion of the (x, y)–plane which lies inside $x^2 + y^2 = 4$ and between the lines $y = 0$ and $y = x$.

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