Integral Calculus – Mth 252

Archive – Spring 1998 Files

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This document contains five sample problem sets, the quiz and final final exam and two notes from Mth 252 Spring 1998. In some cases solutions have been provided. The original test formatting has not been preserved.

Contents

1 Sample Problems Set 1

Problem 1. Evaluate the definite integral

$$\int_0^1 \frac{1-x}{1+x}\, dx$$

Problem 2. Evaluate the definite integral

$$\int_0^1 \frac{1+x}{1+x^2}\,dx$$

Problem 3. Evaluate the definite integral

$$\int_1^2 \frac{x+1}{(x^2+2x+3)^2}\,dx$$

Problem 4. Evaluate the indefinite integral

$$\int (1+x^2)^{-3/2}\,dx$$

by making the substitution $x = \tan(\theta)$.

Problem 5. Evaluate the indefinite integral

$$\int (x + \frac{1}{x})\,dx$$

Problem 6. Evaluate the definite integral

$$\int_0^1 (1+t^2)^{100} t^3\,dt$$

The answer turns out to be

$$\frac{845100400152152934331135470250 67}{6868}$$

so you might want to leave your answer in a more convenient form, rather than simplifying it.

Problem 7. If

$$g(x) = \int_x^{x^2} \arctan(e^x)\,dx$$

calculate the derivative $g'(x)$.

Problem 8. Evaluate the indefinite integral

$$\int \cos^3(x)\,dx$$

Your answer might be $C + \sin(x) - \frac{1}{3}\sin^3(x)$ or perhaps $C + (\frac{2}{3} + \frac{1}{3}\cos^2(x))\sin(x)$. Are these answers the same?

Problem 9. Compute the derivative of

$$x^{\sin(x)}$$

Problem 10. Compute the limit

$$\min_{x \to 0} \frac{x^3}{x - \arctan(x)}$$

(Recall l'Hospital's rule).

---

2 Sample Problems Set 2

```
> restart;
> popt:=`color,noborder`:
```

Problem 11. Compute the volume swept out by rotating the area under the graph of $y = \sec(x)$ above the interval $[0, \frac{\pi}{4}]$ about the x-axis.

```
> `The volume is V = `; Int(Pi*(sec(x)^2),x=0..Pi/4) =
> int(Pi*(sec(x)^2),x=0..Pi/4);
```

$$\mathit{The\ volume\ is\ V\ } =$$

$$\int_0^{1/4\,\pi} \pi \sec(x)^2\, dx = \pi$$

Problem 12. Verify by differentiating that

```
> Int((log(x))^2,x) = int((log(x))^2,x);
```

$$\int \ln(x)^2\, dx = \ln(x)^2\, x - 2\, x \ln(x) + 2\, x$$

Now consider the area under the graph of $y = \log(x)$ from 1 to e. Compute the volume swept out if this area is rotated about the x-axis.

```
> `The volume is V = `; Int(Pi*(log(x)^2),x=1..exp(1)) =
> int(Pi*(log(x)^2),x=1..exp(1));
```

$$\mathit{The\ volume\ is\ V\ } =$$

$$\int_1^e \pi \ln(x)^2\, dx = \pi\, e - 2\,\pi$$

Problem 13. Find the volume swept out if the area in problem 2 is rotated about the y-axis instead.

```
> 'The volume is V = '; Int(Pi*(exp(2)-exp(2*y)),y=0..1) =
> int(Pi*(exp(2)-exp(2*y)),y=0..1);
```

$$\textit{The volume is } V \;=$$

$$\int_0^1 \pi\,(e^2 - e^{(2\,y)})\,dy = \frac{1}{2}\,e^2\,\pi + \frac{1}{2}\,\pi$$

Problem 14. Two circular cylinders of radius a intersect at right angles in such a way that their axes intersect. The volume in common to both cylinders looks like a *pincushion* on elliptical frames. Find the volume of the pincushion.

```
> 'The volume is V = '; Int(4*(a^2-x^2),x=-a..a) =
> int(4*(a^2-x^2),x=-a..a);
```

$$\textit{The volume is } V \;=$$

$$\int_{-a}^{a} 4\,a^2 - 4\,x^2\,dx = \frac{16}{3}\,a^3$$

Problem 15. The area between $y = \cos(x)$ and $y = \sin(x)$ from 0 to $\frac{\pi}{2}$ is rotated about the x-axis. Find the volume swept out.

```
> 'The volume is V = '; Int(Pi*abs((cos(x))^2-(sin(x))^2),x=0..Pi/2) =
> Pi*int(abs((cos(x))^2-(sin(x))^2),x=0..Pi/2);
```

$$\textit{The volume is } V \;=$$

$$\int_0^{1/2\,\pi} \pi\,\left|-\cos(x)^2 + \sin(x)^2\right|\,dx = \pi$$

Problem 16. Verify by differentiation

```
> Int(arctan(x),x) = int(arctan(x),x);
```

$$\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2)$$

Use this result to compute

```
> Int(arctan(x),x=0..1) = int(arctan(x),x=0..1);
```

$$\int_0^1 \arctan(x)\,dx = \frac{1}{4}\,\pi - \frac{1}{2}\ln(2)$$

Problem 17. Find the area of the bounded region bounded by $y = 2\,x^3 + 10\,x + x - 2$ and $y = 3\,x^2 - x + 1$.

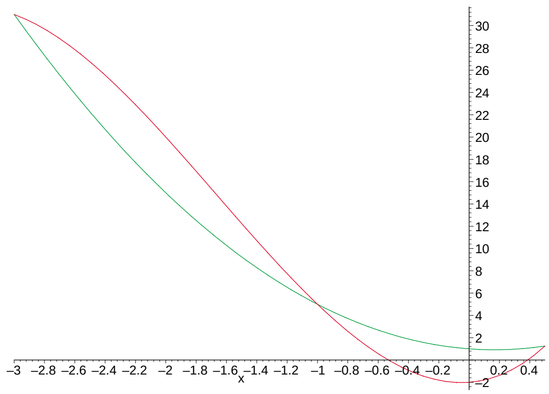
First we find the points of intersection by solving the equations simultaneously:

```
> solve(2*x^3+10*x^2+x-2=3*x^2-x+1,x);
```

$$-3,\ \frac{1}{2},\ -1$$

This solution suggests we look at the graphs from -3 to $\frac{1}{2}$.

```
> pcmd:=[2*x^3+10*x^2+x-2, 3*x^2-x+1],x=-3..1/2:
> pname:=`252s1998_archive_01.eps`:
> plotsetup(ps,plotoutput=pname,plotoptions=popt);
> plot(pcmd);
> plotsetup(default);
> plot(pcmd);
```



Because of the change of sign we either have to divide the interval of integration at -1 or use the absolute value in the integral. We have:

```
> `The area A = `; Int(abs((2*x^3+10*x^2+x-2)-(3*x^2-x+1)),x=-3..1/2) =
> int(abs((2*x^3+10*x^2+x-2)-(3*x^2-x+1)),x=-3..1/2);
```

$$\textit{The area } A =$$

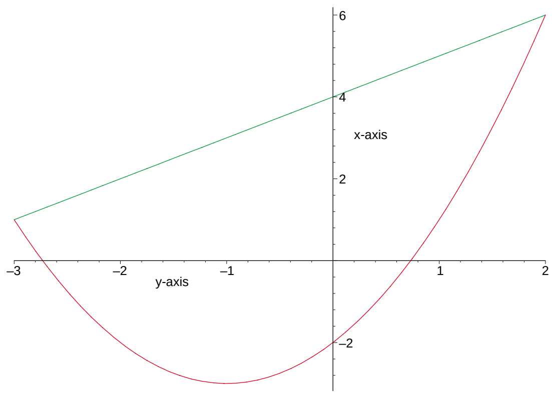
$$\int_{-3}^{1/2} \left|2\,x^3 + 7\,x^2 + 2\,x - 3\right|\,dx = \frac{937}{96}$$

Problem 18. Find the area of the bounded region bounded by the parabola $x = y^2 + 2\,y - 2$ and the line $y = x - 4$.

```
> solve(y^2+2*y-2 = y+4, y);
```

$$-3,\ 2$$

```
> pcmd:=[y^2+2*y-2,y+4],y=-3...2,labels=['y-axis','x-axis']:
> pname:='252s1998_archive_02.eps':
> plotsetup(ps,plotoutput=pname,plotoptions=popt);
> plot(pcmd);
> plotsetup(default);
> plot(pcmd);
```




```
> 'Area A ='; Int((y+4)-(y^2+2*y-2),y=-3..2) = int((y+4)-(y^2+2*y-2),y=-
3..2);
```

$$\mathit{Area\ A}\ =$$

$$\int_{-3}^{2} -y + 6 - y^2\, dy = \frac{125}{6}$$

```
> 'Alternately we can do the following:  '; Solve(x=y^2+2*y-2,y); y = solve(x=
2,y);
```

$$\mathit{Alternately\ we\ can\ do\ the\ following:}$$

$$\mathrm{Solve}(x = y^2 + 2\,y - 2,\, y)$$

$$y = (-1 + \sqrt{x+3},\, -1 - \sqrt{x+3})$$

```
> 'Area A = '; Int( (-1+sqrt(3+x))-(-1-sqrt(3+x)),x=-3..1) + Int((-1+sqrt(3+x
(x-4),x=1..6) = int( (-1+sqrt(3+x))-(-1-sqrt(3+x)),x=-3..1) + int((-1+sqrt(3+x
(x-4),x=1..6);
```

$$\mathit{Area\ A}\ =$$

$$\int_{-3}^{1} 2\sqrt{x+3}\, dx + \int_{1}^{6} 3 + \sqrt{x+3} - x\, dx = \frac{125}{6}$$

3 Sample Problems Set 3

```
> restart;
```

Problem 19.

Find all numbers $0 < b$ such that the average value of $3\,x^2 + 2\,x - 2$ on the interval $[0,\,b]$ is 4.

```
> expr:=1/b*Int(3*x^2+2*x-2,x=0..b);
```

$$expr := \frac{\int_0^b 3\,x^2 + 2\,x - 2\,dx}{b}$$

```
> expr:=value(expr);
```

$$expr := \frac{b^3 + b^2 - 2\,b}{b}$$

```
> solve(expr=4,b);
```

$$-3,\,2$$

Thus the solution is $b = 2$.

Problem 20.

A heavy uniform rope hangs over the edge of a building taller than the length of the rope. It would require 1200 ft-lb of work to pull the entire rope to the top of the building. How much work is done by pulling up just half of the rope to the top of the building?

```
> W0:=Int(x*w,x=0..L); W0:=value(W0); W1:=Int(x*w,x=0..L/2)+(L*w/2)*(L/2);
W1:=value(W1);;
```

$$\mathrm{W0} := \int_0^L x\,w\,dx$$

$$\mathrm{W0} := \frac{1}{2}\,w\,L^2$$

$$\mathrm{W1} := \int_0^{1/2\,L} x\,w\,dx + \frac{1}{4}\,w\,L^2$$

$$\mathrm{W1} := \frac{3}{8}\,w\,L^2$$

```
> Work:=W1*1200/W0;
```

$$Work := 900$$

Problem 21.

Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = \sin(x)$ and $y = 0$ from $x = 0$ to $x = \pi$.

```
> 2*Pi*x*sin(x),x=0..Pi:Int(%)=int(%);
```

$$\int_0^\pi 2\,\pi\,x\sin(x)\,dx = 2\,\pi^2$$

Problem 22.

Find the volume of the solid otained by rotating about the x-axis the region described in problem 3.

```
> Pi*(sin(x))^2,x=0..Pi:Int(%)=int(%);
```

$$\int_0^\pi \pi\sin(x)^2\,dx = \frac{1}{2}\,\pi^2$$

Problem 23.

Evaluate the following integrals. You may wish to use integration by parts, substitution, or both - sometimes several times.

Note your answers may differ from those provided by Maple. In those cases, try to verify that your answer and Maple's answers are equivalent.

```
> x^3*exp(x),x:Int(%)=int(%);
```

$$\int x^3\,e^x\,dx = x^3\,e^x - 3\,x^2\,e^x + 6\,x\,e^x - 6\,e^x$$

```
> x^2*cos(x),x:Int(%)=int(%);
```

$$\int x^2\cos(x)\,dx = x^2\sin(x) - 2\sin(x) + 2\,x\cos(x)$$

```
> exp(2*x)*sin(3*x),x:Int(%)=int(%);
```

$$\int e^{(2\,x)}\sin(3\,x)\,dx = -\frac{3}{13}\,e^{(2\,x)}\cos(3\,x) + \frac{2}{13}\,e^{(2\,x)}\sin(3\,x)$$

```
> x*exp(x)*cos(x),x:Int(%)=int(%);
```

$$\int x\,e^x\cos(x)\,dx = \frac{1}{2}\,x\,e^x\cos(x) - (-\frac{1}{2}\,x + \frac{1}{2})\,e^x\sin(x)$$

```
> x*arctan(x),x:Int(%)=int(%);
```

$$\int x\arctan(x)\,dx = \frac{1}{2}\,x^2\arctan(x) - \frac{1}{2}\,x + \frac{1}{2}\arctan(x)$$

```
> arctan(x),x:Int(%)=int(%);
```

$$\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2)$$

```
> x*log(x),x:Int(%)=int(%);
```

$$\int x\ln(x)\,dx = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2$$

```
> (log(x))^2,x:Int(%)=int(%);
```

$$\int \ln(x)^2\,dx = \ln(x)^2 x - 2x\ln(x) + 2x$$

```
> (log(x))^3,x:Int(%)=int(%);
```

$$\int \ln(x)^3\,dx = \ln(x)^3 x - 3\ln(x)^2 x + 6x\ln(x) - 6x$$

```
> (a+b*x+c*log(x))^3,x:Int(%)=int(%);
```

$$\int (a+bx+c\ln(x))^3\,dx = a^3x + \frac{3}{2}a^2bx^2 + 3ca^2x\ln(x) - 3ca^2x + ab^2x^3$$
$$+ 3cabx^2\ln(x) - \frac{3}{2}cabx^2 + 3c^2a\ln(x)^2x - 6c^2ax\ln(x) + 6c^2ax + \frac{1}{4}b^3x^4$$
$$+ cb^2x^3\ln(x) - \frac{1}{3}cb^2x^3 + \frac{3}{2}c^2bx^2\ln(x)^2 - \frac{3}{2}c^2bx^2\ln(x) + \frac{3}{4}c^2bx^2 + c^3\ln(x)^3x$$
$$- 3c^3\ln(x)^2x + 6c^3x\ln(x) - 6c^3x$$

```
> sqrt(x)*log(x),x:Int(%)=int(%);
```

$$\int \sqrt{x}\ln(x)\,dx = \frac{2}{3}x^{(3/2)}\ln(x) - \frac{4}{9}x^{(3/2)}$$

By doing an integration by parts and a substitution you should be able to reduce the problem of integrating $\sqrt{x}\arctan(x)$ to the problem of integrating a certain rational function. You can then use the method of partial fractions (which we have not yet covered) to finish it of. In case you want to play with it, heer is the result:

```
> sqrt(x)*arctan(x),x:Int(%)=int(%);
```

$$\int \sqrt{x}\arctan(x)\,dx = \frac{2}{3}x^{(3/2)}\arctan(x) - \frac{4}{3}\sqrt{x} + \frac{1}{6}\sqrt{2}\ln(\frac{x+\sqrt{x}\sqrt{2}+1}{x-\sqrt{x}\sqrt{2}+1})$$
$$+ \frac{1}{3}\sqrt{2}\arctan(\sqrt{x}\sqrt{2}+1) + \frac{1}{3}\sqrt{2}\arctan(\sqrt{x}\sqrt{2}-1)$$

```
> (sec(x))^4,x:Int(%)=int(%);
```

$$\int \sec(x)^4\,dx = \frac{1}{3}\frac{\sin(x)}{\cos(x)^3} + \frac{2}{3}\frac{\sin(x)}{\cos(x)}$$

```
> sin(log(x)),x:Int(%)=int(%);
```

$$\int \sin(\ln(x))\,dx = -\frac{1}{2}\cos(\ln(x))\,x + \frac{1}{2}\sin(\ln(x))\,x$$

```
> Int(exp(sqrt(x)),x)=int(exp(sqrt(x)),x);
```

$$\int e^{(\sqrt{x})}\,dx = 2\,e^{(\sqrt{x})}\,\sqrt{x} - 2\,e^{(\sqrt{x})}$$

```
> x^5*sin(x^3),x:Int(%)=int(%);
```

$$\int x^5\sin(x^3)\,dx = \frac{1}{3}\sin(x^3) - \frac{1}{3}\,x^3\cos(x^3)$$

Problem 24. A certain amount of work is done to stretch a spring 3 in. The force to keep it stretched is 16 lb. How much additional work is done to stretch the spring an additional 4 in. (Use lb-in for the units of work in this problem).

```
> dx:=3:  df:=16:  k:=df/dx:
> work:=Int(k*x,x=3..3+4); work:=value(work);
```

$$\mathit{work} := \int_3^7 \frac{16}{3}\,x\,dx$$

$$\mathit{work} := \frac{320}{3}$$

Problem 25. Compute the limit as x goes to 1 of

```
> 1/(x-1)*Int(exp(t)*log(t+1),t=1..x);
```

$$\frac{\int_1^x e^t\ln(t+1)\,dt}{x-1}$$

You should get $e\log(2)$, but you will not have much luck if you try to evaluate the integral.

---

4 Sample Problems Set 4

I have (reluctantly) provided answers for some of the problems below. Please note in some cases I may have mistyped the answer. Whether or not your answer agrees with mine, you should check it.

Problem 26. Find the area of the surface generated by rotating the curve $y = x^3$, $0 \le x \le b$, about the x–axis.

Answer: $\frac{\pi}{27}\left[\left(1+9b^4\right)^{3/2} - 1\right]$.

Problem 27. Find the area of the surface generated by rotating the curve $y = \mathrm{e}^x$, $0 \le x \le \frac{3}{2}\log(x)$, about the x–axis.

Answer: $\pi\left[5\sqrt{2}+\log\left(\frac{2\sqrt{2}+3}{\sqrt{2}+1}\right)\right]$.

Problem 28. Find the area of the surface generated by rotating the curve $y=x^2$, $0\le x\le\sqrt{2}$, about the x–axis.

Answer: $\frac{\pi}{32}\left[102\sqrt{2}+\log\left(3-2\sqrt{2}\right)\right]$.

Problem 29. Find the area of the surface generated by rotating the curve $\frac{x^2-\log(x)}{\sqrt{8}}$, $1\le x\le b$, about the x–axis.

Answer: $\frac{\pi}{8}\left(b^4+2b^2-\log^2(b)-2b^2\log(b)-3\right)$.

Problem 30. Find the length of the curve $y=\log(\cos x)$, $0\le x\le\frac{\pi}{6}$.

Answer: $\frac{1}{2}\log 3$.

Problem 31. Find the length of the curve $y=\frac{x^3}{6}+\frac{1}{2x}$, $1\le x\le b$.

Answer: $\frac{b^4+2b-3}{6b}$.

Problem 32. Find the length of the curve $y=\frac{1}{2}x^2$, $0\le x\le b$.

Answer: $\frac{1}{2}b\sqrt{1+b^2}+\frac{1}{2}\log\left(b+\sqrt{1+b^2}\right)$.

The problems below are cribbed from Mth 256 tests and review sheets. In Mth 256 a number of techniques for solving differential equations are discussed. In Mth 252 we have studied only one technique, separation of variables, which works only for separable equations. Hopefully I have selected only problems involving separable equations below. Have fun!

Problem 33. Consider a tank of volume V which is full of, for example, water. Assume water runs into the tank at a rate of α liters/minute and drains out of an opening in the bottom at the same rate. At a certain time t_0 we inject Q_0 grams of dye into the inflow. Assume that the dye instantly enters the tank and is uniformly well–mixed throughout the tank at all successive times. If Q is the amount of dye in the tank at time t then

$$\frac{dQ}{dt} = -\alpha\,\frac{Q}{V}, \quad Q(t_0) = Q_0.$$

At a certain time $t_1 > t_0$ the concentration of dye in the outflow is found to be β grams/liter. Another measurement 3.5 minutes later is found to yield a concentration of $\beta/2$ grams/liter. If $\alpha = 2$ liters/minute find the volume of the tank.

Here a few separable first order ordinary differential equations to test your skill:

Problem 34.

$$\frac{dy}{dt} = 4 + y^2$$

Problem 35.

$$\frac{dy}{dx} = \cos y$$

Problem 36.

$$\frac{dy}{dt} = y^2 - 3y + 2$$

Problem 37.

$$\frac{dy}{dx} = y^2 - 2y + 1$$

Problem 38.

$$\frac{dx}{dt} = \frac{t^2}{x}$$

Problem 39.

$$\frac{dy}{dx} = \frac{x^2}{y(1 + x^3)}$$

The solutions to the problems above are one–parameter solutions, that is, contain an arbitrary constant. Sometimes we have an extra condition the solution must satisfy, for example, an initial condition. In that case we have to determine a value of the parameter so the corresponding solution satisfies the extra condition. Here are some examples:

Problem 40.

$$\frac{dy}{dx} = \frac{xy + y}{x}, \quad y(1) = -2.$$

Problem 41.

$$\frac{dy}{dx} = \frac{xy + y}{x} \log(x), \quad y(1) = -2.$$

Here $\log x$ means the *natural* logarithm of x. Assume $x > 0$.

Problem 42.

$$\frac{dy}{dx} + 2xy = 0, \quad y(0) = -2$$

Problem 43. Solve the initial value problem

$$\frac{dy}{dx} = \frac{y}{x} \log(x), \quad y(\mathrm{e}) = \mathrm{e}$$

where e is the base of the natural logarithm (EULER's number) and $\log x$ means the *natural* logarithm of x. Assume $x > 0$.

Problem 44.

$$x\frac{dy}{dx} + y = 0, \quad y(1) = 4$$

Problem 45.

$$x\frac{dy}{dx} - 4y = 0, \quad y(2) = -8$$

Problem 46.

$$\frac{dy}{dt} + (\tan t)y = 0, \quad y(0) = 2$$

Problem 47.

$$\frac{dy}{dt} + \frac{y}{t} = 0, \quad y(1) = -3$$

Problem 48.

$$x^2\frac{dy}{dx} + y = 0, \quad y(1) = 1$$

Problem 49.

$$\frac{dy}{dt} - \frac{y}{t} = 0, \quad y(1) = -3$$

Problem 50.

$$\frac{dy}{dx} + 4x^3 y = 0, \quad y(1) = 2$$

Problem 51.

$$\frac{dy}{dx} + (\log x)y = 0, \quad y(2) = 1$$

Here $\log x$ means the *natural* logarithm of x. Assume $x > 0$.

Problem 52.

$$x\frac{dy}{dx} + (x+1)y = 0, \quad y(2) = \mathrm{e}^{-2}$$

Problem 53.

$$(x^2+1)\frac{dy}{dx} - xy = 0, \quad y(\sqrt{3}) = -1$$

Problem 54.

$$(x^2+4)\frac{dy}{dx} + y = 0, \quad y(\pi/2) = \sqrt{e}$$

Problem 55.

$$\frac{dy}{dx} + (\cos x)y = 0, \quad y(0) = 2$$

Problem 56.

$$\frac{dy}{dx} + (3\sin x)y = 0, \quad y(0) = 1/2$$

Problem 57.

$$\frac{dy}{dx} + (6x^2 + 2x)y = 0, \quad y(1) = 1$$

Problem 58.

$$\frac{dy}{dx} - (\cot x)y = 0, \quad y(\pi/2) = 1$$

Problem 59.

$$\frac{dy}{dx} + y\cos(x) = \cos(x), \quad y(0) = 3.$$

Problem 60. Solve the initial value problem

$$\frac{dp}{dt} = \mathrm{e}^{-2t}\,p(1-p), \quad p(0) = 0.5.$$

Find

$$\lim_{t\to\infty} p(t).$$

Problem 61. A radioactive substance decays to 85% of its original mass in 36 hours. Find the half–life. Find the third–life.

NEWTON's law of cooling (or heating) states that when a body at temperature T is immersed in a medium at temperature A then the rate of change of T is proportional to the difference in the temperatures. Explicitly

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant.

NEWTON's law of cooling is a useful application of first order ordinary differential equations.

Problem 62. A cup of coffee initially at temperature $T_0 = 190°$ F is brought into a room at temperature $A = 65°$ F. The heat capacity of the room (compared to the coffee) is so large that we may regard A as being constant. After 2 minutes the temperature of the coffee is $145°$ F. What temperature will the coffee be an additional minute later.

Problem 63. A thermometer reading $92°$ F is immersed in a cooler liquid. After 3 seconds the thermometer reads $80°$ F. Another 3 seconds later it reads $76°$ F. What is the temperature of the fluid?

Problem 64. A thermometer initially reading $62°$ F is placed in a well insulated cup of very hot coffee. After 2 seconds the thermometer reads $167°$ F. After an additional 1 second it reads $179°$ F. If A denotes the temperature of the coffee, T denotes the temperature reading of the thermometer and t denotes time in seconds then according to NEWTON

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant. We regard the temperature A of the coffee also as constant. Find the temperature of the coffee.

Problem 65. A thermometer is brought into a certain room. The room has temperature $A = 25°$ C. If T is the temperature displayed by the thermometer then according to NEWTON

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant. After being in the room for 10 seconds the thermometer reads $21.4°$ C. An additional 20 seconds later it reads $23.4°$ C. What was the initial reading on the thermometer at the time that it was first brought into the room?

Problem 66. Consider a cup of coffee in a room of temperature A. Initially the temperature of the coffee is $183°$ F, 3 minutes later the temperature is $155°$ F, and an additional 3 minutes later the temperature is $135°$ F. Find the temperature T as a function of time. Compute the temperature of the room.

---

5 Sample Problems Set 5

Here are some sample integrals. In most cases I have provided answers. These integrals illustrate many of the techniques that we have studied. Some of them are quite difficult – other are easy. You can not tell which is which by looking at them. So dig in.

I have not provided any sample problems concerning numerical quadrature. Nonetheless you should have a good understanding of Simpson's rule, at the very least.

Please keep in mind there may be typos in the answers.

Problem 67. Evaluate

$$\int \frac{\tan x + \sin x}{\sec x}\, dx$$

Ans: $C - \cos x - \dfrac{1}{2}\cos^2 x$

Problem 68. Evaluate

$$\int \frac{\sin^3 x}{\sqrt{\cos x}}\, dx.$$

Ans: $C - 2\cos^{1/2} x + \dfrac{2}{5}\cos^{5/2} x$

Problem 69. Evaluate

$$\int \frac{\sec^4\theta}{\tan^2\theta}\,d\theta.$$

Ans: $C + \tan\theta - \cot\theta = C + \dfrac{1}{\cos\theta\,\sin\theta} - 2\dfrac{\cos\theta}{\sin\theta}$

Problem 70. Evaluate

$$\int \frac{x^{11}}{(1+x^4)^{5/2}}\,dx.$$

Ans: $\mathbb{C} + \dfrac{3x^8 + 12x^4 + 8}{6\,(1+x^4)^{3/2}} = C + \dfrac{1}{6}\,\dfrac{3x^8 + 6x^4 + 2}{(1+x^4)^{3/2}} + \dfrac{1}{\sqrt{1+x^4}}$

Problem 71. Evaluate

$$\int \frac{\log x}{x^3}\,dx.$$

Ans: $C - \dfrac{2\log x + 1}{4x^2}.$

Problem 72. Evaluate

$$\int \frac{\cos\theta}{\sin^2\theta + \sin\theta - 6}\,d\theta.$$

Ans: $C + \dfrac{1}{5}\log\left(\dfrac{\sin(\theta) - 2}{\sin(\theta) + 3}\right).$

Problem 73. Evaluate

$$\int \frac{\sec^2\theta}{\tan^3\theta - \tan^2\theta}\,d\theta$$

Ans: $\log(1 - \cot\theta) + \cot\theta$

Problem 74. The integral

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx$$

can be evaluated by making one of the substitutions $x = \sin\theta$, $u = 1 - x^2$ or $w^2 = 1 - x^2$. Try all three substitutions and compare your results.

Problem 75. Evaluate the integral

$$\int x^2 \sqrt{x^2+1}\, dx$$

by making the substitution $x = \tan\theta$. In the resulting integral express everything in terms of $\sec\theta$ and integrate by parts, keeping an eye out for a needed devious trick.

Ans: $C + \frac{1}{4}x\left(1+x^2\right)^{3/2} - \frac{1}{8}x\left(1+x^2\right)^{1/2} - \frac{1}{8}\log\left(x + \left(1+x^2\right)^{1/2}\right)$

Problem 76. Evaluate the integral

$$\int \frac{x^2}{\sqrt{x^2+1}}\, dx$$

by making the substitution $x = \tan\theta$. In the resulting integral express everything in terms of $\sec\theta$ and integrate by parts, keeping an eye out for a needed devious trick. This problem is easier than the previous one.

Ans: $C + \frac{1}{2}x\sqrt{1+x^2} - \frac{1}{2}\log\left(x + \sqrt{1+x^2}\right)$

Problem 77. Evaluate the integral

$$\int \frac{x^3}{\sqrt{x^2+1}}\, dx$$

by making the substitution $w^2 = 1 + x^2$. Try also to do it by integration by parts, for example, with $u = x^2$ and $dv = x\sqrt{1+x^2}dx$. You should also try the substitution $x = \tan\theta$. Compare all of your answers.

Ans: $C + \frac{1}{5}\left(1+x^2\right)^{5/2} - \frac{1}{3}\left(1+x^2\right)^{3/2}$

Problem 78. Evaluate

$$\int \frac{1}{x^2+4x+5}\, dx.$$

Ans: $C + \arctan(x+2)$

Problem 79. Evaluate

$$\int \frac{1}{(x^2+4x+5)^2}\, dx.$$

Ans: $C + \frac{1}{2}\frac{x+2}{x^2+4x+5} + \frac{1}{2}\arctan(x+2)$

Problem 80. Evaluate
$$\int \frac{3x^3 + 2x - 3}{(x^2 + 6x + 25)^2}\, dx.$$

Ans: $C + \frac{1}{16}\frac{171x + 233}{x^2 + 6x + 25} - \frac{261}{64}\arctan\left(\frac{x+3}{4}\right) + \frac{3}{2}\log\left(x^2 + 6x + 25\right)$

Problem 81. Evaluate
$$\int x\sqrt{1+x}\, dx.$$

Ans: $C - \frac{2}{3}(1+x)^{3/2} + \frac{2}{5}(1+x)^{5/2}$

Problem 82. Evaluate
$$\int (\mathrm{e}^x - 1)^{1/2}\ dx$$

Ans: $C + 2\sqrt{\mathrm{e}^x - 1} - 2\arctan\left(\sqrt{\mathrm{e}^x - 1}\right)$

And now a message from your sponsor $\cdots$

Why evaluate integrals when computers can do it?

Computers are pretty good at doing grubby and tedious chores – people are not. It would be nice if we could concentrate on the high-level stuff and let the computer handle the details. Things are never so neat though. Understanding is reluctant to visit us and frequently requires hard, grubby, detail work first. You practice doing integrals so you can learn how things work and so you can understand arguments based on integration. It's true that the details of integrating some combination of sines and cosines will not stay with you and will probably not be important to you – after all the computer can do it – but the pain and sweat will give you insight and understanding that you are unlikely to come by any other way. The basic ideas and the major techniques – *substitution, integration by parts, partial fractions* – are important, should be mastered, and will be mastered, through hard work and hard thinking, not by punching a keyboard.

6 Quiz

Problem 83. Find the area of the region bounded by $y = \sin(x)$ and $y = \frac{2x}{\pi}$ for $0 \leq x \leq \frac{\pi}{2}$.

Problem 84. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = x^3$ for $0 \le x \le 1$ about the x-axis.

Problem 85. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = x^3$ for $0 \le x \le 1$ about the y-axis.

Problem 86. Compute the derivative

$$\frac{d}{dx}\int_0^{\sqrt{x}} \mathrm{e}^{t^2}\,dt$$

Problem 87. If a certain amount of work W is required to stretch a spring by b feet from its equilibrium length, how much additional work is required to stretch it by an additional b feet?

Problem 88. Find the antiderivative (indefinite integral, primitive)

$$\int \arctan\left(\sqrt{t}\right)\,dt$$

by first making the substitution $t = x^2$ and then integrating by parts. Your answer should contain an arbitrary constant (constant of integration) and should be expressed in terms of t.

---

7 Quiz Solutions (Maple)

```
> restart;
> `Problem 1`; int(sin(x)-2*x/Pi,x=0..Pi/2);
```

Problem 1

$$-\frac{1}{4}\pi + 1$$

```
> `Problem 2`; int(Pi*((x^2)^2-(x^3)^2),x=0..1);
```

Problem 2

$$\frac{2}{35}\pi$$

```
> `Problem 3`; int(2*Pi*x*(x^2-x^3),x=0..1);
```

Problem 3

$$\frac{1}{10}\pi$$

```
> 'Problem 4'; diff(int(exp(t^2),t=0..sqrt(x)),x);
```

$$\textit{Problem 4}$$

$$\frac{1}{2}\frac{e^x}{\sqrt{x}}$$

```
> 'Problem 5'; W:=int(k*x,x=0..b); W1:=int(k*x,x=b..2*b); 'W1'=W1/W*'W';
```

$$\textit{Problem 5}$$

$$\mathrm{W} := \frac{1}{2}k\,b^2$$

$$\mathrm{W1} := \frac{3}{2}k\,b^2$$

$$\mathrm{W1} = 3\,\mathrm{W}$$

```
> 'Problem 6'; int(arctan(sqrt(t)),t);
```

$$\textit{Problem 6}$$

$$t\arctan(\sqrt{t}) - \sqrt{t} + \arctan(\sqrt{t})$$

8 Final Exam

Problem 89. A thermometer intially reading 19.6° C is brought into a certain room. The room has temperature $A = 25.0^\circ$ C. If T is the temperature displayed by the thermometer then according to NEWTON

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant. After being in the room for 10 seconds the thermometer reads 21.4° C. What will it read an additional 20 seconds later?

Problem 90. Compute the derivative $\dfrac{d}{dx}\displaystyle\int_0^{\sqrt{x}} \mathrm{e}^{t^2}\,dt$.

① e^x

② $\mathrm{e}^x\sqrt{x}$

③ $\dfrac{\mathrm{e}^x}{\sqrt{x}}$

④ $\dfrac{\mathrm{e}^x}{x}$

⑤ None of the above.

☐ ←*Letter corresponding to your answer to problem 90.*

Problem 91. Let $b > 1$. Suppose $x = a > 1$ is a solution of the equation $x^{\log x} = b$. Find another solution.

① $\log a$
② $-a$
③ $\frac{1}{a}$
④ $1 - a$
⑤ None of the above.

☐ ←*Letter corresponding to your answer to problem 91.*

Problem 92. Find the arc length of the curve $y = x^{3/2}, \quad 0 \le x \le 5$.

① $\frac{325}{27}$
② $\frac{335}{27}$
③ $\frac{343}{27}$
④ $6\sqrt{5}$
⑤ $5\sqrt{6}$
⑥ None of the above.

☐ ←*Letter corresponding to your answer to problem 92.*

Problem 93. Find the area of the surface generated by rotating the curve $y = x^2$, $0 \le x \le \sqrt{2}$ about the y-axis.

Problem 94. For a certain function f we have

$f(0.50) = 0.64$	$f(0.75) = 0.75$	$f(1.00) = 1.02$	$f(1.25) = 1.10$
$f(1.50) = 1.15$	$f(1.75) = 1.05$	$f(2.00) = 0.80$	$f(2.25) = 0.65$

Use Simpson's rule and some of the data above to estimate the integral

$$\int_{1.00}^{2.00} f(x)\,dx.$$

Problem 95. Evaluate the integral $\int e^{\sqrt{x}}\,dx$.

① $C + e^{\sqrt{x}+1}/\left(\sqrt{x}+1\right)$
② $C + xe^{\sqrt{x}} - e^{\sqrt{x}}$
③ $C + \sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}$
④ $C + 2\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}$
⑤ None of the above.

☐ ←*Letter corresponding to your answer to problem 95.*

Problem 96. Evaluate the integral $\int \frac{2x+7}{x^2-x-6}\,dx$.

Problem 97. Evaluate the integral $\int \frac{x-7}{x^2+6x+10}\,dx$.

Problem 98. To evaluate an integral involving the expression $\left(x^2+4x+13\right)^{1/2}$ one of the following substitutions is frequently helpful:

① $x = -2 + 3\sec\theta$
② $x = -3 + 2\sec\theta$
③ $x = -2 + \tan\theta$
④ $x = -2 + 3\tan\theta$
⑤ $x = -3 + 2\tan\theta$
⑥ $x = -3 + 2\sin\theta$
⑦ None of the above.

☐ ←*Letter corresponding to your answer to problem 98.*

---

9 Partial Fractions in Maple

The Maple convert command provides a convenient way to expand rational functions in partial fractions. Here are some examples:

Example 1

```
> f15:=(x^3+x^2+1)/(x^4+x^3+2*x^2);
```

$$f15 := \frac{x^3 + x^2 + 1}{x^4 + x^3 + 2\,x^2}$$

```
> convert(f15,parfrac,x);
```

$$\frac{1}{2}\,\frac{1}{x^2} - \frac{1}{4}\,\frac{1}{x} + \frac{1}{4}\,\frac{3 + 5\,x}{x^2 + x + 2}$$

Example 2

```
> f3:=(x^2+3*x-4)/((2*x-1)^2*(3*x+3));
```

$$f3 := \frac{x^2 + 3\,x - 4}{(2\,x - 1)^2\,(3\,x + 3)}$$

```
> convert(f3,parfrac,x);
```

$$-\frac{1}{2}\,\frac{1}{(2\,x - 1)^2} + \frac{11}{18}\,\frac{1}{2\,x - 1} - \frac{2}{9}\,\frac{1}{x + 1}$$

Example 3

```
> g1:=(x^3-2*x^2+3*x+5)/((x^2+1)*(x^2-1));
```

$$g1 := \frac{x^3 - 2\,x^2 + 3\,x + 5}{(x^2 + 1)\,(x^2 - 1)}$$

```
> convert(g1,parfrac,x);
```

$$\frac{7}{4}\,\frac{1}{x - 1} + \frac{1}{4}\,\frac{1}{x + 1} - \frac{1}{2}\,\frac{7 + 2\,x}{x^2 + 1}$$

Example 4

```
> f50:=(x^3-2*x^2+x+1)/(x^4+5*x^2+4);
```

$$f50 := \frac{x^3 - 2\,x^2 + x + 1}{x^4 + 5\,x^2 + 4}$$

```
> convert(f50,parfrac,x);
```

$$\frac{x - 3}{x^2 + 4} + \frac{1}{x^2 + 1}$$

Example 5

```
> f67:=(4*x^3-27*x^2+5*x-32)/(30*x^5-13*x^4+50*x^3-286*x^2-299*x-70);
```

$$f67 := \frac{4\,x^3 - 27\,x^2 + 5\,x - 32}{30\,x^5 - 13\,x^4 + 50\,x^3 - 286\,x^2 - 299\,x - 70}$$

```
> convert(f67,parfrac,x);
```

$$\frac{24110}{4879}\,\frac{1}{5\,x + 2} - \frac{668}{323}\,\frac{1}{2\,x + 1} - \frac{9438}{80155}\,\frac{1}{3\,x - 7} + \frac{1}{260015}\,\frac{48935 + 22098\,x}{x^2 + x + 5}$$

Example 6

```
> f68:=(12*x^5-7*x^3-13*x^2+8)/(100*x^6-80*x^5+116*x^4-80*x^3+41*x^2-20*x+4);
```

$$f68 := \frac{12\,x^5 - 7\,x^3 - 13\,x^2 + 8}{100\,x^6 - 80\,x^5 + 116\,x^4 - 80\,x^3 + 41\,x^2 - 20\,x + 4}$$

```
> convert(f68,parfrac,x);
```

$$\frac{5828}{1815}\,\frac{1}{(5\,x-2)^2} - \frac{59096}{19965}\,\frac{1}{5\,x-2} + \frac{2}{3993}\,\frac{816+2843\,x}{2\,x^2+1} + \frac{1}{363}\,\frac{-251+313\,x}{(2\,x^2+1)^2}$$

10 Bernoulli Sums

Examples of sums of powers of integers (Bernoulli sums).

```
> Sum(k^2,k=1..n):  % = collect(value(%),n);
```

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}\,n^3 + \frac{1}{2}\,n^2 + \frac{1}{6}\,n$$

```
> Sum(k^3,k=1..n):  % = collect(value(%),n);
```

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}\,n^4 + \frac{1}{2}\,n^3 + \frac{1}{4}\,n^2$$

```
> Sum(k^4,k=1..n):  % = collect(value(%),n);
```

$$\sum_{k=1}^{n} k^4 = \frac{1}{5}\,n^5 + \frac{1}{2}\,n^4 + \frac{1}{3}\,n^3 - \frac{1}{30}\,n$$

```
> Sum(k^5,k=1..n):  % = collect(value(%),n);
```

$$\sum_{k=1}^{n} k^5 = \frac{1}{6}\,n^6 + \frac{1}{2}\,n^5 + \frac{5}{12}\,n^4 - \frac{1}{12}\,n^2$$

```
> Sum(k^13,k=1..n):  % = collect(value(%),n);
```

$$\sum_{k=1}^{n} k^{13} = \frac{1}{14}\,n^{14} + \frac{1}{2}\,n^{13} + \frac{13}{12}\,n^{12} - \frac{143}{60}\,n^{10} + \frac{143}{28}\,n^8 - \frac{143}{20}\,n^6 + \frac{65}{12}\,n^4 - \frac{691}{420}\,n^2$$

It is begining to look like

```
> Sum(k^m,k=1..n) = (1/(m+1)*n^(m+1))+1/2*n^m+O(n^(m-1));
```

$$\sum_{k=1}^{n} k^m = \frac{n^{(m+1)}}{m+1} + \frac{1}{2}\,n^m + \mathrm{O}(n^{(m-1)})$$

PROBLEM: Can you prove this fact (perhaps by induction)?

The coefficients that occur in these sums are related to the Bernoulli numbers (in case you want to search for more information). The sums of powers of successive integers were studied by Jakob Bernoulli (1665-1705). He found the formulae above for powers through the 10th and investigated the numbers now named after him.

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