Integral Calculus – Mth 252

Archive – Fall 1993 Files

July 9, 1999)

This document contains three sets of sample problems, two quizzes, the final exam, and a note from Mth 252 Fall 1993. For a very few of the problems I have provided answers. The original test formatting has not been preserved.

Contents

1 Sample Problems - Set 1

Problem 1. Given

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

compute

$$\sum_{j=1}^{n} (6j+1)^2.$$

Problem 2. Compute the derivative

$$\frac{d}{dx}\int_0^{x^2} \sin\left(t^2\right)\, dt.$$

Problem 3. Find the average value of $\sin(x)$ on the interval $[0, \pi]$.

Problem 4. Evaluate the integral

$$\int_0^2 \left| x - x^3 \right| \, dx.$$

Problem 5. Evaluate the integral

$$\int_0^2 \left(1 + x^3\right)^{3/2} x^2 \, dx.$$

Problem 6. Compute the indefinite integral

$$\int \frac{\cos(x)}{\sqrt{1+\sin(x)}}\, dx = C + \ldots$$

Problem 7. Find the area of the bounded region bounded by the graphs of $y = x^2$ and $y = \sqrt{x}$.

Problem 8. Find the area of the bounded region bounded by the graphs of $y = x$ and $x = 2 - y^2$.

Problem 9. Find the volume of the solid generated by rotating about the x–axis the bounded region bounded by the curves $y = x^3$ and $y = x^2$.

Problem 10. Find the volume of the solid generated by rotating about the y–axis the bounded region bounded by the curves $y = x^3$ and $y = x^2$.

Problem 11. A (Hookean) spring has a natural length of 2 ft. and a force of 50 lbs. is required to stretch it by $1/2$ ft. How much work is done in stretching the spring from its natural length to a length of 4 ft.

Problem 12. Compute the derivative

$$\frac{d}{d\theta} \log\left(\cos(\theta)\right).$$

2 Quiz 1

Problem 13. Compute

$$\sum_{k=1}^{n} \big((k+1)^3 - k^3\big).$$

Hint: You may use

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \quad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

if you wish, but there is a better way.

Problem 14. Find the average value of $y = \left| 4 - x^2 \right|$ on the interval $[-3, 4]$.

Problem 15. Find the area of the bounded region bounded by the graphs of $y = -x + 4$ and $x = 2 + y^2$.

Problem 16. Evaluate (exactly) the definite integrals

$$\textbf{(A)} \quad \int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx \qquad\qquad \textbf{(B)} \quad \int_0^1 \frac{x^3}{\sqrt{1+x^2}}\,dx.$$

Problem 17. A (Hookean) spring has a natural length of 0.50 m. A force of 8.00 N compresses the spring by 0.02 m. How much work would be done in stretching the spring from its natural length to a length of 0.75 m. (N = newton, m = meter, N-m (joule) is the corresponding unit of work).

Problem 18. Find the volume of the solid generated by rotating about the x-axis the bounded region bounded by the curves $y = -x^2 + 4x$ and $y = x$.

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3 Sample Problems - Set 2

Problem 19. Find *all positive* solutions of $x^{x-1} = \left(x^2\right)^3$.

Answer 19. $x = 1$ or $x = 7$.

Problem 20. Find $\frac{dy}{dx}$ if $y = \mathrm{e}^{x^2} x \log(x)$.

Answer 20. $2\,x^2\,e^{x^2}\log(x) + e^{x^2}\log(x) + e^{x^2}$.

Problem 21. A certain radioactive substance has a half-life of 4621 hours. How long does it take to decay to $3/4$*th* of the original amount? For this problem use the approximations $\log 2 \approx 0.69315$ and $\log 3 \approx 1.09861$ and give your answer to 4 significant digits.

Answer 21. 1918 hours.

Problem 22. What fraction of a radioactive substance decays in one half of the half-life? Give your answer to 4 significant figures.

Answer 22. 29.29 %.

Problem 23. Evaluate

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx \quad (x > 0).$$

Answer 23. $c + x - 2\sqrt{x} + \log(x + 2\sqrt{x} + 1)$

Problem 24. Evaluate

$$\int x\,e^{x^2+e^{x^2}}\,dx.$$

Answer 24. $c + \frac{1}{2}\,e^{e^{x^2}}$

Problem 25. A thermometer initially reading $62°$ F is placed in a well insulated cup of very hot coffee. After 2 seconds the thermometer reads $167°$ F. After an additional 1 second it reads $179°$ F. If A denotes the temperature of the coffee, T denotes the temperature reading of the thermometer and t denotes time in seconds then according to Newton

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant. We regard the temperature A of the coffee also as constant. Find the temperature of the coffee.

Answer 25. You should get $e^{-k} = \frac{14}{35}$ and then be able to show $A = 187°$ F.

The quiz will also cover derivatives and integrals of trigonometric functions and inverse trigonometric functions, as well as hyperbolic functions and integration by parts.

Problem 26. Evaluate

$$\int \sin^3(2x)\,\cos^2(2x)\,dx.$$

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4 Quiz 2

Problem 27. Find all solutions $x > 0$ of the equation

$$x^x \, \mathrm{e}^{x(\log(x))^2} = x^{3+x\log(x)}.$$

Problem 28. Evaluate the integral

$$\int \frac{\mathrm{e}^{2x}}{1+\mathrm{e}^x} \, dx.$$

Simplify your answer as much as possible by using properties of exponentials and logarithms.

Problem 29. A cup of very hot coffee is brought into a room. The temperature of the room is $71°$F. At a certain time the coffee is determined to have temperature $196°$F. The temperature of the coffee is measured every 5 minutes after this initial time. The first measurement yields $171°$F. What temperatures should the next two measurements yield?

In this problem if A denotes the (essentially constant) temperature of the room, T denotes the temperature of the coffee and t is time then Newton's law of cooling yields

$$\frac{dT}{dt} = -k(T - A)$$

where k is a constant (depending on the insulating properties of the cup).

Problem 30.

(A) What fraction of a radioactive substance decays in twice the half-life?

(B) What fraction of a radioactive substance decays in two-thirds of the half life? Give your answer to 4 significant figures.

You might find the following natural logarithms useful

$$\log(2) = 0.6931471 \qquad \log(3) = 1.0986123$$

Problem 31. Use the substitution $u = \sec(x)$ to evaluate the integral

$$\int \tan^3(x) \, \sec(x) \, dx.$$

Problem 32. Evaluate the integral

$$\int \sin^2(3x)\ \cos^3(3x)\, dx.$$

Problem 33. Integrate by parts

$$\int x^2\ \cos(2x)\, dx.$$

5 Sample Problems - Set 3

Problem 34. Evaluate

$$\int \sin^4(2\,x)\ \cos^2(2\,x)\, dx.$$

Answer 34.

$$C - \frac{\sin^3(2\,x)\cos^3(2\,x)}{12} - \frac{\sin(2\,x)\cos^3(2\,x)}{16} + \frac{\cos(2\,x)\sin(2\,x)}{32} + \frac{x}{16}$$

Problem 35.

$$\int \frac{1}{(1+\sin(x))^2}\, dx$$

Answer 35.

$$C - \frac{4}{3\,\left(\tan(\frac{x}{2})+1\right)^3} + \frac{2}{\left(\tan(\frac{x}{2})+1\right)^2} - \frac{2}{\tan(\frac{x}{2})+1}$$

Problem 36.

$$\int \frac{\sqrt[4]{x+1}}{x}\, dx$$

Answer 36.

$$C + 4\,\sqrt[4]{x+1} + \log(\sqrt[4]{x+1} - 1) - \log(\sqrt[4]{x+1} + 1) - 2\ \arctan(\sqrt[4]{x+1})$$

Problem 37. (Text p. 450 #6)

$$\int x^2\ (x-1)^{3/2}\ dx$$

Answer 37.

$$C+\frac{2\ (x-1)^{9/2}}{9}+\frac{2\ (x-1)^{5/2}}{5}+\frac{4\ (x-1)^{7/2}}{7}$$

Problem 38. (Text p. 450 #4)

$$\int \frac{1}{x^{1/2}-x^{1/4}}\, dx$$

Answer 38.

$$C+2\sqrt{x}+\log(\frac{-1-x+2\sqrt{x}}{x-1})+4\sqrt[4]{x}+2\ \log(\sqrt[4]{x}-1)-2\ \log(\sqrt[4]{x}+1)+\log(x-1)$$

Problem 39. (Text p. 450 #12)

$$\int \sqrt{\frac{1+x}{1-x}}\, dx$$

Answer 39.

$$C+\sqrt{1-x^2}-\arcsin(x)$$

Problem 40. (Text p. 453 #60)

$$\int \sin\left(\sqrt{x}\right)\ dx$$

Answer 40.

$$C+2\sin\left(\sqrt{x}\right)-2\sqrt{x}\cos\left(\sqrt{x}\right)$$

Problem 41. (Text p. 453 #78)

$$\int \sqrt{1+\cos(x)}\, dx$$

Answer 41.

$$C + 2\sqrt{2}\,\sin\left(\frac{x}{2}\right), \qquad -\pi \le x \le \pi$$

Problem 42. (Text p. 453 #79 - Maple chokes on this one, but you can do it by first computing $(\cos(\phi) + \sin(\phi))^2$, simplifying and then looking hard at the result. You can also use the substitution discussed in section 9.8 p.448, but that's harder. Perhaps the easiest way to do it is by multiplying and dividing the integrand by $\sqrt{1-\sin(x)}$. This last, and easiest, method yields an answer in the same form as the one in the text.)

$$\int \sqrt{1+\sin(x)}\,dx$$

Answer 42.

$$C + 2\sin\left(\frac{x}{2}\right) - 2\cos\left(\frac{x}{2}\right) = C - 2\left(1-\sin(x)\right)^{1/2}$$

The first expression is correct for $-\pi/2 \le x \le 3\pi/2$ whereas the second expression is correct for $-\pi/2 \le x \le \pi/2$.

Problem 43. (Text p. 453 #94)

$$\int \log\left(1+\sqrt{x}\right)\,dx$$

Answer 43.

$$C + \left(1+\sqrt{x}\right)^2 \log\left(1+\sqrt{x}\right) + \sqrt{x} - x/2 - 2\left(1+\sqrt{x}\right)\log\left(1+\sqrt{x}\right)$$

Problem 44. (Text p. 453 #100)

$$\int x\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)^{1/2}\,dx$$

Answer 44.

$$C + \frac{\sqrt{1-x^4}}{2} + \frac{\arcsin(x^2)}{2}$$

Problem 45. Find the total area of the four-leaved "rose" $r = a\,\sin(2\theta)$.

Answer 45.

$$\frac{\pi a^2}{2}$$

Problem 46.

$$\int \frac{\sin(x)}{1+\sin(x)}\,dx$$

Answer 46.

$$C + \frac{2}{1+\tan(\frac{x}{2})} + 2\ \arctan(\tan(\frac{x}{2})) = C + \sec(x) - \tan(x) + x$$

The first expression is the answer you get when you use the substitution $u = \tan(x/2)$ as in section 9.8. The second expression you get by first multiplying and dividing the integrand by $1 - \sin(x)$ and then simplifying.

Problem 47.

$$\int \frac{x}{x^2 + 4\,x + 13}\,dx$$

Answer 47.

$$C + \frac{\log(x^2 + 4\,x + 13)}{2} - \frac{2\ \arctan(\frac{x}{3} + 2/3)}{3}$$

Problem 48.

$$\int \frac{x}{(x^2 + 4\,x + 13)^2}\,dx$$

Answer 48.

$$C + \frac{-2\,x - 13}{18\ (x^2 + 4\,x + 13)} - \frac{\arctan(\frac{x}{3} + 2/3)}{27}$$

Problem 49. (Text p. 440 #20)

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx$$

Answer 49.

$$C + \frac{\sqrt{1+x^2}\,x^2}{3} - \frac{2\,\sqrt{1+x^2}}{3}$$

Note a trigonometric substitution will work but there is an easier way.

Problem 50.

$$\int \frac{x^2+1}{x^3 - x^2 - 2\,x}\,dx$$

Answer 50.

$$C - \frac{\log(x)}{2} + \frac{2\,\log(x+1)}{3} + \frac{5\,\log(x-2)}{6}$$

Problem 51. (Text p. 425 #28)

$$\int x\,\arctan(x)\,dx$$

Answer 51.

$$C + \frac{x^2\arctan(x)}{2} - \frac{x}{2} + \frac{\arctan(x)}{2}$$

Problem 52. (Text p. 425 #22)

$$\int \log\left(1+x^2\right)\,dx$$

Answer 52.

$$C + x\,\log(1+x^2) - 2\,x + 2\,\arctan(x)$$

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6 Exam

Problem 53. If

$$f(x) = \int_0^{x^2+3\,x+1} \sin\left(t^3\right)\,dt$$

compute the derivative $f'(x)$ of $f(x)$. **Note:** Do not attempt to evaluate the integral.

Problem 54. Let A be the region between the curves $y = 1$ and $y = 1 + \sin(x)$ for $x \in [0, \pi]$. Find the volume of the solid that is generated when A is rotated about the x–axis.

Problem 55. The half–life of radioactive cobalt is 5.27 years. Suppose that a nuclear accident left the level of cobalt radiation in a certain place at 64 times the safe level. How long will it take for the radiation level from the radioactive cobalt to fall to the safe level?

Problem 56. Find the area inside the circle $x^2 + y^2 = \frac{1}{2}$ and above the parabola $y = 2\,x^2$.

Problem 57. Use the trigonometric substitution $\theta = \arcsin(x)$ to evaluate the indefinite integral

$$\int \sqrt{1 - x^2}\, dx$$

Problem 58. Evaluate the indefinite integral

$$\int \frac{x^2 + 1}{x^3 - x^2 - 2\,x}\, dx$$

Problem 59. Evaluate the indefinite integral

$$\int \frac{x - 1}{\left(x^2 + 2\,x + 5\right)^2}\, dx.$$

Problem 60. Make the substitution $x = u^3$ to evaluate the integral

$$\int \frac{dx}{x + x^{1/3}}$$

Problem 61. Use integration by parts to evaluate the indefinite integral

$$\int \log\left(1 + x^2\right)\, dx$$

Problem 62. Find the area of *one* leaf of the *three-leaved rose*

$$r = 2\,a\,\cos(3\,\theta)$$

where $a > 0$ is a constant (and r, θ denote polar coordinates as usual).

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7 Note on a Certain Integral

Here's the solution to problem 55, section 9.4 (page 426) in our text. As usual $\log(x)$ means the natural logarithm of x. Maple does the problem in a fraction of a second. Can you do it in less than an hour? Why would an engineer need three weeks to do it and then not be sure that it's correct?

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> restart; p:=(k*log(x)-2*x^3+3*x^2+b)^4:  Int(p,x) = sort(int(p,x),x);
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$$\int (k\ln(x) - 2x^3 + 3x^2 + b)^4\,dx = \frac{16}{13}x^{13} - 8x^{12} + \frac{216}{11}x^{11} - \frac{16}{5}k\ln(x)x^{10} + \frac{8}{25}kx^{10}$$
$$- \frac{16}{5}bx^{10} - \frac{108}{5}x^{10} - \frac{16}{9}kx^9 + 16k\ln(x)x^9 + 9x^9 + 16bx^9 - 27bx^8 + \frac{27}{8}kx^8$$
$$- 27k\ln(x)x^8 + \frac{24}{7}b^2x^7 + \frac{48}{7}kb\ln(x)x^7 + \frac{108}{7}bx^7 - \frac{48}{49}kbx^7 + \frac{24}{7}k^2\ln(x)^2x^7$$
$$- \frac{108}{49}kx^7 - \frac{48}{49}k^2\ln(x)x^7 + \frac{108}{7}k\ln(x)x^7 + \frac{48}{343}k^2x^7 - 12k^2\ln(x)^2x^6 - \frac{2}{3}k^2x^6$$
$$- 12b^2x^6 - 24k\ln(x)bx^6 + 4k^2\ln(x)x^6 + 4kbx^6 - \frac{108}{25}kbx^5 + \frac{54}{5}b^2x^5$$
$$- \frac{108}{25}k^2\ln(x)x^5 + \frac{54}{5}k^2\ln(x)^2x^5 + \frac{108}{125}k^2x^5 + \frac{108}{5}k\ln(x)bx^5 - 2b^3x^4 + \frac{3}{16}k^3x^4$$
$$- 6kb^2\ln(x)x^4 - \frac{3}{4}k^3\ln(x)x^4 + 3k^2b\ln(x)x^4 - 2k^3\ln(x)^3x^4 + \frac{3}{2}k^3\ln(x)^2x^4$$
$$+ \frac{3}{2}kb^2x^4 - \frac{3}{4}k^2bx^4 - 6k^2b\ln(x)^2x^4 + 12k^2\ln(x)^2bx^3 - 4kb^2x^3 + \frac{8}{3}k^2bx^3$$
$$+ 4k^3\ln(x)^3x^3 + 4b^3x^3 + 12k\ln(x)b^2x^3 - 4k^3\ln(x)^2x^3 - 8k^2b\ln(x)x^3 - \frac{8}{9}k^3x^3$$
$$+ \frac{8}{3}k^3\ln(x)x^3 + 24k^3b\ln(x)x + 4k^3b\ln(x)^3x - 12k^3b\ln(x)^2x + 12k^2b^2x$$
$$+ 24k^4x + 4kb^3\ln(x)x - 24k^4\ln(x)x - 4kb^3x - 24k^3bx - 4k^4\ln(x)^3x + b^4x$$
$$+ k^4\ln(x)^4x - 12k^2b^2\ln(x)x + 12k^4\ln(x)^2x + 6k^2b^2\ln(x)^2x$$

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