

# Discrete Mathematics – Mth 232

Archive – Winter 2000 Files

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This archive contains the problems from a sample quiz, two quizzes and the final exam from Mth 232 Spring 2000. The original test instructions, headers and formatting have not been preserved. In addition some extra problems are included and there is a brief note on calculating with floor and ceiling functions.

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## 1 Sample Test-Problems

**Problem 1.** A countable union of countably infinite sets is

- A.) Finite
- B.) Countable
- C.) Uncountable
- D.) In one-to-one correspondence with the points in the interval  $(0, 1)$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 1).

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**Problem 2.** The set of all sequences  $(a_1, a_2, a_3, \dots)$  where  $a_k = 0$  or  $a_k = 1$  for each  $k$  is

- A.) Finite
- B.) Countable
- C.) Uncountable
- D.) Blue
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 2).

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**Problem 3.** The set of all sequences  $(a_1, a_2, a_3, \dots)$  where  $a_k = 0$  or  $a_k = 1$  for each  $k$ , and  $a_k = 0$  for  $k$  sufficiently large (depending on the sequence) is

- A.) Finite
- B.) Countable
- C.) Uncountable
- D.) Blue
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 3).

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**Problem 4.** Consider the set  $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ . Which of the following statements is true?

- A.) If we select any 6 numbers from  $S$  at least 2 of them have sum 15.
- B.) If we select any 7 numbers from  $S$  at least 2 of them have sum 15.
- C.) If we select any 6 numbers from  $S$  at least 2 of them have sum 14.
- D.) If we select any 7 numbers from  $S$  at least 2 of them have sum 14.
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 4).

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**Problem 5.** You select  $n$  integers at random. How big *must*  $n$  be in order to guarantee that 5 of the selected integers have the same remainder when divided by 7?

- A.) 21
- B.) 22
- C.) 28
- D.) 29
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 5).

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**Problem 6.** Four algorithms for solving a particular problem are known to have the following orders. Which algorithm is fastest for all (sufficiently) large  $n$ ?

- A.)  $\mathcal{O}(n^{1.001})$
- B.)  $\mathcal{O}(n(\log n)^2)$
- C.)  $\mathcal{O}(n^2)$
- D.)  $\mathcal{O}(2^n)$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 6).

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**Problem 7.** If

$$f(x) = \frac{3x^{3/2} - 2x + x^{1/2}}{x^{3/4} + 4x^{5/4} - x}$$

then the *best* order estimate on  $f$  is

- A.)  $f(x)$  is  $\mathcal{O}(x^{3/4})$
- B.)  $f(x)$  is  $\mathcal{O}(x^{-5/4})$
- C.)  $f(x)$  is  $\mathcal{O}(x^{1/4})$
- D.)  $f(x)$  is  $\mathcal{O}(x^{3/2})$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 7).

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**Problem 8.** The maximum number of edges for a simple graph with 20 vertices is

- A.) 20
- B.) 40
- C.) 190
- D.) 380
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 8).

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**Problem 9.** A certain graph has vertices with degrees 3, 4, 5, 6 and 6. How many edges does it have?

- A.) 5
- B.) 10
- C.) 12
- D.) 24
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 9).

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**Problem 10.** The  $n$ -cube,  $n \geq 1$ , is a simple graph with  $2^n$  vertices. Each vertex has degree  $n$ . How many edges does the  $n$ -cube have?

- A.)  $2^n$
- B.)  $n2^n$
- C.)  $2^{n-1}$
- D.)  $n2^{n-1}$
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 10).

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**Problem 11.** The number of 4-tuples of integers  $i, j, k, l$  with

$$1 \leq i \leq j \leq k \leq l \leq 12$$

is:

- A.) 495
- B.) 11880
- C.) 1365
- D.) 715
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 11).

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**Problem 12.** The number of 4-tuples of integers  $i, j, k, l$  with

$$1 \leq i < j < k < l \leq 12$$

is:

- A.) 495
- B.) 11880
- C.) 1365
- D.) 715
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 12).

---

**Problem 13.** The number of 4-tuples of integers  $i, j, k, l$  with

$$1 \leq i < j < k \leq l \leq 12$$

is:

- A.) 495
- B.) 11880
- C.) 1365
- D.) 715
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 13).

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**Problem 14.** How many numbers must you choose from  $\{3, 4, 5, 6, 7, 8\}$  in order to ensure that some 3 of them have sum 15?

- A.) 3
- B.) 4
- C.) 5
- D.) 6
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 14).

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## 2 Quiz 1

**Problem 15.** A fair coin is tossed 10 times. What is the probability of getting 5 or fewer heads?

- A.)  $\frac{1}{4}$
- B.)  $\frac{1}{3}$
- C.)  $\frac{1}{2}$
- D.)  $\frac{3}{4}$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 15).

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**Problem 16.** A fair coin is tossed 9 times. What is the probability of getting 4 or fewer heads?

- A.)  $\frac{1}{4}$
- B.)  $\frac{1}{3}$
- C.)  $\frac{1}{2}$
- D.)  $\frac{4}{9}$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 16).

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**Problem 17.** The number of integers from 29 to 1032 divisible by 3 but not by 5 is

- A.) 335
- B.) 268
- C.) 267
- D.) 67
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 17).

---

**Problem 18.** How many 3-permutations are there of 7 distinguishable objects?

- A.) 6
- B.) 24
- C.) 210
- D.) 5040
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 18).

---

**Problem 19.** Four women and four men are to be seated in a row with men and women alternating. How many ways can they be seated?

- A.) 40320
- B.) 2340
- C.) 1152
- D.) 576
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 19).

---

**Problem 20.** How many four (decimal) digit numbers have no repeated digits?

- A.) 3024
- B.) 5040
- C.) 4516
- D.) 4536
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 20).

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**Problem 21.** There are  $5^{10} = 9,765,625$  strings of length 10 containing just the characters  $\{a,b,c,d,e\}$ . How many of these strings contain a pair of consecutive characters that are the same?

- A.) 5,242,880
- B.) 4,522,754
- C.) 1,310,720
- D.) 8,454,905
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 21).

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**Problem 22.** How many integers from 37 to 9342 are multiples of 5 or 7?

- A.) 265
- B.) 1861
- C.) 2925
- D.) 3190
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 22).

---

**Problem 23.** How many permutations of ABCDE have A, C or E as the first letter and C, D or E as the second letter?

- A.) 42
- B.) 54
- C.) 72
- D.) 84
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 23).

---

**Problem 24.** How many ways can 3 distinguishable objects be chosen from 7 distinguishable objects with no regard for order?

- A.) 6
- B.) 35
- C.) 210
- D.) 840
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 24).

---

**Problem 25.** A committee of 6 is to be chosen from 9 women and 7 men. The committee must contain at least 3 women. How many possible committees are there?

- A.) 6552
- B.) 6468
- C.) 2940
- D.) 6553
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 25).

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**Problem 26.** We have 3 red marbles, 4 blue marbles and 2 black marbles. The marbles can be distinguished only by color. How many distinct linear arrangements are possible?

- A.) 1260
- B.) 288
- C.) 362880
- D.) 1420
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 26).

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**Problem 27.** How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 37$$

with each  $x_j \geq 2$  an integer?

- A.) 7140
- B.) 9880
- C.) 4960
- D.) 8240
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 27).

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**Problem 28.** Let  $X = \{a, b, c\}$  and let  $Y = \{d, e, f, g\}$ . A function  $f : X \rightarrow Y$  is chosen at random. What is the probability that  $f : X \rightarrow Y$  is injective (one-to-one)?

- A.) 1/8
- B.) 3/8
- C.) 1/2
- D.) 3/4
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 28).

---

**Problem 29.** Let  $X = \{a, b, c, d, e\}$  and let  $Y = \{1, 2, 3\}$ . How many surjective (onto) functions  $f: X \rightarrow Y$  are there?

- A.) 96
- B.) 147
- C.) 150
- D.) 243
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 29).

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**Problem 30.** A deck of playing cards contains 4 suits. Each suit contains 13 cards. If you choose cards at random, how many *must* you choose to be sure of getting at least 5 cards from the same suit?

- A.) 13
- B.) 16
- C.) 17
- D.) 18
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 30).

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### 3 Quiz 2

**Problem 31.** A countable union of countable sets is

- A.) Finite
- B.) Countable
- C.) Uncountable
- D.) It is impossible to tell
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 31).

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**Problem 32.** The set of all sequences  $(a_1, a_2, a_3, \dots)$  where  $a_k = 0$  or  $a_k = 1$  for each  $k$  is

- A.) Finite
- B.) Countable
- C.) Uncountable
- D.) It is impossible to tell
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 32).

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**Problem 33.** If  $f(x)$  is known to be  $\mathcal{O}(x^{3/4}(\log x)^5)$  then it is also

- A.)  $\mathcal{O}(x^{3/4}(\log x)^4)$
- B.)  $\mathcal{O}(x^{2/3}(\log x)^8)$
- C.)  $\mathcal{O}(x^{5/6}(\log x)^2)$
- D.)  $\mathcal{O}(x^{3/4})$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 33).

---

**Problem 34.** Let  $K_n$  be the complete graph with  $n$  vertices, that is  $K_n$  is a simple graph with  $n$  vertices and with any two vertices adjacent. Recall an Euler circuit is a closed path containing each edge exactly once and each vertex at least once. For what values of  $n$  does  $K_n$  have an Euler circuit?

- A.)  $n$  even
- B.)  $n$  odd
- C.)  $n$  a power of 2
- D.) Never
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 34).

---

**Problem 35.** A certain graph has vertices with degrees 5, 7, 6, 6 and 6. How many edges does it have?

- A.) 5
- B.) 10
- C.) 12
- D.) 24
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 35).

---

**Problem 36.** The number of 4-tuples of integers  $i, j, k, l$  with

$$1 \leq i \leq j < k \leq l \leq 12$$

is:

- A.) 495
- B.) 715
- C.) 1001
- D.) 1365
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 36).

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**Problem 37.** For any set  $X$  let  $\mathfrak{P}(X)$  be the power set of  $X$ , that is, the set of all subsets of  $X$ . Let  $h: X \rightarrow \mathfrak{P}(X)$  be any function and let  $A = \{x \in X \mid x \notin h(x)\}$ .

- A.)  $A$  is in the range of  $h$
- B.)  $A$  is not in the range of  $h$
- C.)  $A = X$  (always)
- D.)  $A$  is not well-defined. There is a paradox here.
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 37).

---

**Problem 38.** You select  $n$  integers at random. How big *must*  $n$  be in order to guarantee that 4 of the selected integers have the same remainder when divided by 7?

- A.) 21
- B.) 22
- C.) 28
- D.) 29
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 38).

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**Problem 39.** The graph with adjacency matrix

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 2 & 1 \end{bmatrix}$$

has how many edges?

- A.) 5
- B.) 13
- C.) 14
- D.) 23
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 39).

---

**Problem 40.** Suppose a sort algorithm runs in  $\mathcal{O}(n \log_{10} n)$  time to sort  $n$  items. If it takes 0.108 seconds to sort 1,000 items what is your best estimate for the length of time to sort 100,000 items?

- A.) 10.8 seconds
- B.) 14.6 seconds
- C.) 18.0 seconds
- D.) 21.4 seconds
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 40).

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**Problem 41.** Suppose a sort algorithm runs in  $\mathcal{O}(n^2)$  time to sort  $n$  items. If it takes 0.108 seconds to sort 1,000 items what is your best estimate for the length of time to sort 100,000 items?

- A.) 1080 seconds
- B.) 1460 seconds
- C.) 1800 seconds
- D.) 2140 seconds
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 41).

**Problem 42.** How many distinct graphs are there with 28,830 vertices with degrees as follows

No. vertices	Degree of each vertex	No. vertices	Degree of each vertex
5376	29	4763	37
4722	31	3985	39
4337	35	5647	42

- A.) 57,660
- B.) 122,347
- C.) 104
- D.) 33
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 42).

## 4 Final Exam

**Problem 43.** How many integers from (including) 81 to (including) 1079 are divisible by 3 but not by 9?

- A.) 216
- B.) 222
- C.) 312
- D.) 333
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 43).

**Problem 44.** Let  $A$  be a finite-state automaton with input alphabet  $I = \{a, b\}$  and states  $S = \{s_0, s_1, s_2, s_3, s_4\}$ . Assume that  $s_0$  is the initial state and that  $s_3$  is the only accepting state. Let the next-state table of  $A$  be

	$a$	$b$
$\rightarrow s_0$	$s_1$	$s_2$
$s_1$	$s_1$	$s_3$
$s_2$	$s_4$	$s_2$
$\odot s_3$	$s_4$	$s_2$
$s_4$	$s_1$	$s_3$

Which of the following strings is accepted by  $A$ :

- A.) ababa
- B.) abbbb
- C.) babab
- D.) aaabb
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 44).

**Problem 45.** How many ways are there to arrange 9 indistinguishable balls into 5 distinguishable buckets?

- A.) 70
- B.) 175
- C.) 350
- D.) 715
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 45).

**Problem 46.** How many ways are there to arrange 9 indistinguishable balls into 5 distinguishable buckets, if no bucket is to remain empty?

- A.) 70
- B.) 175
- C.) 350
- D.) 715
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 46).

**Problem 47.** How many ways are there to arrange 9 indistinguishable balls into 5 distinguishable buckets, if some bucket must contain at least 5 balls?

- A.) 70
- B.) 175
- C.) 350
- D.) 715
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 47).

---

**Problem 48.** How many ways are there to arrange 5 indistinguishable balls into 5 indistinguishable buckets?

- A.) 1
- B.) 7
- C.) 11
- D.) 126
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 48).

---

**Problem 49.** How many solutions are there to the equation  $x_1 + x_2 + x_3 + x_4 = 127$  if each  $x_k \geq 0$  is an integer?

- A.) 333375
- B.) 349504
- C.) 357760
- D.) 366175
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 49).

---

**Problem 50.** Choose the *best* order estimate for  $x \sin(x)$  from among the following:

- A.)  $\mathcal{O}(1)$
- B.)  $\mathcal{O}(x)$
- C.)  $\mathcal{O}(x^2)$
- D.)  $\mathcal{O}(\sin(x))$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 50).

---

**Problem 51.** If

$$\frac{3x^7 - 4x^5 + x + 3}{x^4 + 1} - 3x^3$$

choose the *best* order estimate for  $f(x)$  from among the following:

- A.)  $\mathcal{O}(1)$
- B.)  $\mathcal{O}(x)$
- C.)  $\mathcal{O}(x^2)$
- D.)  $\mathcal{O}(x^3)$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 51).

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**Problem 52.** Let the graph  $G$  consist of the edges and vertices of an ordinary cube. Then  $G$  has 8 vertices and 12 edges and all the vertices have the same degree. Which of the following statements is true for  $G$ .

- A.)  $G$  has an Euler circuit
- B.)  $G$  has a Hamiltonian circuit
- C.)  $G$  has a spanning tree with 8 edges
- D.)  $G$  has no spanning tree
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 52).

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**Problem 53.** A countable union of countably infinite sets is

- A.) Finite
- B.) Countable
- C.) Uncountable
- D.) It depends on the sets
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 53).

---

**Problem 54.** Let  $A$  be the  $2 \times 2$  matrix  $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ . Then  $A$  is invertible and the inverse of  $A$  is of the form  $A^{-1} = \begin{bmatrix} -1 & x-2 \\ x & -2 \end{bmatrix}$ . Find  $x$ .

- A.) 1
- B.) 2
- C.) 3
- D.) 4
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 54).

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**Problem 55.** You note that the worst time estimate for finding an item in a sorted array of 1,000,000 data items of a certain type using binary search is about 12 seconds. What is your best estimate for the worst-case length of time required to search a sorted array of 10,000,000 items of the same type?

- A.) 14 seconds
- B.) 15 seconds
- C.) 24 seconds
- D.) 120 seconds
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 55).

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**Problem 56.** Consider the graph  $G$  described as follows:  $G$  has four vertices  $v_1, v_2, v_3, v_4$  and four edges  $a, b, c, d$  – moreover  $a$  is incident on  $v_1$  and on  $v_2$ ,  $b$  is incident on  $v_2$  and on  $v_3$ ,  $c$  is incident on  $v_1$  and on  $v_3$ , and  $d$  is incident on  $v_3$  and on  $v_4$ . How many spanning trees does  $G$  have? (Do not count isomorphic trees as identical unless they are identical subgraphs.)

- A.) 1
- B.) 2
- C.) 3
- D.) 4
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 56).

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**Problem 57.** Consider the simple graph  $G$  described as follows:  $G$  has four vertices  $v_1, v_2, v_3, v_4$  – moreover  $v_1$  and  $v_2$  are adjacent,  $v_2$  and  $v_3$  are adjacent,  $v_1$  and  $v_3$  are adjacent, and  $v_3$  and  $v_4$  are adjacent. How many walks of length 4 from  $v_4$  to  $v_3$  are there?

- A.) 1
- B.) 2
- C.) 3
- D.) 4
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 57).

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**Problem 58.** Let  $G$  be a graph with  $n$  vertices. Then the number of vertices of odd degree is

- A.) odd
- B.) even
- C.) 0
- D.)  $\lfloor n/2 \rfloor$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 58).

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**Problem 59.** An Euler circuit is a closed walk containing

- A.) each edge at most once and each vertex exactly once
- B.) each edge at least once and each vertex exactly once
- C.) each edge at exactly once and each vertex at least once
- D.) each edge at least once and each vertex at least once
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 59).

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**Problem 60.** A Hamiltonian circuit is a closed walk containing

- A.) each edge at most once and each vertex exactly once
- B.) each edge at least once and each vertex exactly once
- C.) each edge at exactly once and each vertex at least once
- D.) each edge at least once and each vertex at least once
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 60).

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**Problem 61.** Let  $T$  be a tree with  $n \geq 2$  vertices. Then the total degree of  $T$  is

- A.)  $2n$
- B.)  $2n - 1$
- C.)  $2n - 2$
- D.)  $2n + 1$
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 61).

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**Problem 62.** A connected graph with  $n$  vertices and  $n$  edges, for any integer  $n \geq 2$ , always has

- A.) a circuit
- B.) an Euler circuit
- C.) a Hamiltonian circuit
- D.) an isolated vertex
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 62).

**Problem 63.** How many integers from (including) 81 to (including) 1079 are divisible by 3 but not by 9?

- A.) 216
- B.) 222
- C.) 312
- D.) 333
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 63).

**Problem 64.** Let  $A$  be a finite-state automaton with input alphabet  $I = \{a, b\}$  and states  $S = \{s_0, s_1, s_2, s_3, s_4\}$ . Assume that  $s_0$  is the initial state and that  $s_3$  is the only accepting state. Let the next-state table of  $A$  be

		$a$	$b$
→	$s_0$	$s_1$	$s_2$
	$s_1$	$s_1$	$s_3$
	$s_2$	$s_4$	$s_2$
⊙	$s_3$	$s_4$	$s_2$
	$s_4$	$s_1$	$s_3$

Which of the following strings is accepted by  $A$ :

- A.) ababa
- B.) abbbb
- C.) babab
- D.) aaabb
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 64).

**Problem 65.** How many ways are there to arrange 9 indistinguishable balls into 5 distinguishable buckets?

- A.) 70
- B.) 175
- C.) 350
- D.) 715
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 65).

---

**Problem 66.** How many ways are there to arrange 9 indistinguishable balls into 5 distinguishable buckets, if no bucket is to remain empty?

- A.) 70
- B.) 175
- C.) 350
- D.) 715
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 66).

---

**Problem 67.** How many ways are there to arrange 9 indistinguishable balls into 5 distinguishable buckets, if some bucket must contain at least 5 balls?

- A.) 70
- B.) 175
- C.) 350
- D.) 715
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 67).

---

**Problem 68.** How many ways are there to arrange 5 indistinguishable balls into 5 indistinguishable buckets?

- A.) 1
- B.) 7
- C.) 11
- D.) 126
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 68).

---

**Problem 69.** How many solutions are there to the equation  $x_1 + x_2 + x_3 + x_4 = 127$  if each  $x_k \geq 0$  is an integer?

- A.) 333375
- B.) 349504
- C.) 357760
- D.) 366175
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 69).

---

**Problem 70.** Choose the *best* order estimate for  $x \sin(x)$  from among the following:

- A.)  $\mathcal{O}(1)$
- B.)  $\mathcal{O}(x)$
- C.)  $\mathcal{O}(x^2)$
- D.)  $\mathcal{O}(\sin(x))$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 70).

---

**Problem 71.** If

$$\frac{3x^7 - 4x^5 + x + 3}{x^4 + 1} - 3x^3$$

choose the *best* order estimate for  $f(x)$  from among the following:

- A.)  $\mathcal{O}(1)$
- B.)  $\mathcal{O}(x)$
- C.)  $\mathcal{O}(x^2)$
- D.)  $\mathcal{O}(x^3)$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 71).

---

**Problem 72.** Let the graph  $G$  consist of the edges and vertices of an ordinary cube. Then  $G$  has 8 vertices and 12 edges and all the vertices have the same degree. Which of the following statements is true for  $G$ .

- A.)  $G$  has an Euler circuit
- B.)  $G$  has a Hamiltonian circuit
- C.)  $G$  has a spanning tree with 8 edges
- D.)  $G$  has no spanning tree
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 72).

---

**Problem 73.** A countable union of countably infinite sets is

- A.) Finite
- B.) Countable
- C.) Uncountable
- D.) It depends on the sets
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 73).

---

**Problem 74.** Let  $A$  be the  $2 \times 2$  matrix  $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ . Then  $A$  is invertible and the inverse of  $A$  is of the form  $A^{-1} = \begin{bmatrix} -1 & x-2 \\ x & -2 \end{bmatrix}$ . Find  $x$ .

- A.) 1
- B.) 2
- C.) 3
- D.) 4
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 74).

---

**Problem 75.** You note that the worst time estimate for finding an item in a sorted array of 1,000,000 data items of a certain type using binary search is about 12 seconds. What is your best estimate for the worst-case length of time required to search a sorted array of 10,000,000 items of the same type?

- A.) 14 seconds
- B.) 15 seconds
- C.) 24 seconds
- D.) 120 seconds
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 75).

---

**Problem 76.** Consider the graph  $G$  described as follows:  $G$  has four vertices  $v_1, v_2, v_3, v_4$  and four edges  $a, b, c, d$  – moreover  $a$  is incident on  $v_1$  and on  $v_2$ ,  $b$  is incident on  $v_2$  and on  $v_3$ ,  $c$  is incident on  $v_1$  and on  $v_3$ , and  $d$  is incident on  $v_3$  and on  $v_4$ . How many spanning trees does  $G$  have? (Do not count isomorphic trees as identical unless they are identical subgraphs.)

- A.) 1
- B.) 2
- C.) 3
- D.) 4
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 76).

---

**Problem 77.** Consider the simple graph  $G$  described as follows:  $G$  has four vertices  $v_1, v_2, v_3, v_4$  – moreover  $v_1$  and  $v_2$  are adjacent,  $v_2$  and  $v_3$  are adjacent,  $v_1$  and  $v_3$  are adjacent, and  $v_3$  and  $v_4$  are adjacent. How many walks of length 4 from  $v_4$  to  $v_3$  are there?

- A.) 1
- B.) 2
- C.) 3
- D.) 4
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 77).

---

**Problem 78.** Let  $G$  be a graph with  $n$  vertices. Then the number of vertices of odd degree is

- A.) odd
- B.) even
- C.) 0
- D.)  $\lfloor n/2 \rfloor$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 78).

---

**Problem 79.** An Euler circuit is a closed walk containing

- A.) each edge at most once and each vertex exactly once
- B.) each edge at least once and each vertex exactly once
- C.) each edge at exactly once and each vertex at least once
- D.) each edge at least once and each vertex at least once
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 79).

---

**Problem 80.** A Hamiltonian circuit is a closed walk containing

- A.) each edge at most once and each vertex exactly once
- B.) each edge at least once and each vertex exactly once
- C.) each edge at exactly once and each vertex at least once
- D.) each edge at least once and each vertex at least once
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 80).

---

**Problem 81.** Let  $T$  be a tree with  $n \geq 2$  vertices. Then the total degree of  $T$  is

- A.)  $2n$
- B.)  $2n - 1$
- C.)  $2n - 2$
- D.)  $2n + 1$
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 81).

---

**Problem 82.** A connected graph with  $n$  vertices and  $n$  edges, for any integer  $n \geq 2$ , always has

- A.) a circuit
- B.) an Euler circuit
- C.) a Hamiltonian circuit
- D.) an isolated vertex
- E.) None of the above.

←Write letter corresponding to your answer here and mark it on the scantron (Problem 82).

---

## 5 Additional Problems

These problems were made up for the tests but not used. They were not checked carefully, so watch out for errors.

**Problem 83.** A fair coin is tossed 10 times. What is the probability of getting exactly 5 heads?

- A.)  $\frac{63}{256}$
- B.)  $\frac{64}{256}$
- C.)  $\frac{65}{256}$
- D.)  $\frac{1}{2}$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 83).

---

**Problem 84.** How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 37$$

with each  $x_j \geq 0$  an integer?

- A.) 7140
- B.) 9880
- C.) 4960
- D.) 8240
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 84).

---

**Problem 85.** We have 6 blue marbles and 6 red marbles. We place 4 of the blue marbles and 3 of the red ones in one urn. The remaining 2 blue marbles and 3 red ones are placed in another indistinguishable urn. Next we select one urn at random, and then draw out a randomly chosen marble. What is the probability that it is blue?

- A.)  $\frac{3}{7}$
- B.)  $\frac{17}{35}$
- C.)  $\frac{1}{2}$
- D.)  $\frac{19}{35}$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 85).

---

**Problem 86.** A *bit string* is a finite sequence of 0's and 1's. The total number of 0's and 1's is called the length. How many bit strings of length 10 are there beginning with 101 and ending with 0?

- A.) 8
- B.) 64
- C.) 512
- D.) 1024
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 86).

---

**Problem 87.** A fair coin is tossed 10 times. What is the probability of getting 8 or more heads?

- A.)  $\frac{45}{1024}$
- B.)  $\frac{8}{10}$
- C.)  $\frac{7}{128}$
- D.)  $\frac{1}{5}$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 87).

---

**Problem 88.** Let  $n$  be a positive integer. In counting the factors of  $n$  we include 1 and  $n$  as factors. Suppose  $n = p_1 p_2 \cdots p_m$  where  $p_1, p_2, \dots, p_m$  are distinct primes. How many distinct factors does  $n$  have?

- A.)  $m$
- B.)  $n$
- C.)  $2^m$
- D.)  $2^n$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 88).

---

**Problem 89.** Let  $p$  and  $q$  be prime numbers with  $p \neq q$ . How many integers  $n$  are there with  $1 \leq n \leq pq$  which have a factor of  $p$  or  $q$ ?

- A.)  $p + q - 1$
- B.)  $p + q$
- C.)  $p + q + 1$
- D.)  $pq$
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 89).

---

**Problem 90.** How many four (decimal) digit numbers contain the digit 7 at least once and the digit 9 at least once?

- A.) 920
- B.) 2248
- C.) 3168
- D.) 5416
- E.) None of the above.

← Write letter corresponding to your answer here and mark it on the scantron (Problem 90).

---

## 6 Floor and Roof (Ceiling) Functions

This note illustrates how to carry out an argument involving the floor and roof functions.

If  $x$  is any real number we denote by  $\lfloor x \rfloor$  the *floor* of  $x$ , that is, the largest integer no bigger than  $x$ . Formally  $k = \lfloor x \rfloor$  is defined as the unique integer  $k$  such that

$$k \leq x < k + 1.$$

Mathematicians commonly call  $\lfloor x \rfloor$  the *greatest integer in  $x$* .

If  $x$  is any real number we denote by  $\lceil x \rceil$  the *roof* of  $x$ , that is, the smallest integer no smaller than  $x$ . Formally  $k = \lceil x \rceil$  is defined as the unique integer  $k$  such that

$$k - 1 < x \leq k.$$

Sometimes  $\lceil x \rceil$  is referred to as the *least integer above  $x$*  or the *ceiling* of  $x$ .

The floor of  $x$  is also denoted by  $\lfloor x \rfloor$  since that is easier to type and *floor* is used far more often than *roof*. Note carefully the precise inequalities above.

Here are a few examples:

$$\begin{aligned} \lceil 3 \rceil &= 3 & \lfloor 3 \rfloor &= 3 & \lceil 3.2 \rceil &= 4 & \lfloor 3.2 \rfloor &= 3 \\ \lceil -2.1 \rceil &= -2 & \lfloor -2.1 \rfloor &= -3 & \lceil -0.9 \rceil &= 0 & \lfloor -0.1 \rfloor &= -1 \end{aligned}$$

Let  $m \leq n$  be integers and let  $q \geq 1$  be an integer. In class we saw that the number  $K$  of multiples of  $q$  from  $m$  to  $n$  is given by

$$K = \left\lfloor \frac{n}{q} \right\rfloor - \left\lfloor \frac{m}{q} \right\rfloor + 1.$$

This formula for  $K$  is a bit awkward since it uses both the floor function and the roof function. In the present case, we can find a nicer formula which uses only the floor function.

**Lemma 6.1.** *If  $m$  and  $q$  are integers,  $q \geq 1$ , then*

$$\left\lceil \frac{m}{q} \right\rceil - 1 = \left\lfloor \frac{m-1}{q} \right\rfloor.$$

*Proof.* Let

$$h = \left\lfloor \frac{m-1}{q} \right\rfloor.$$

This means that  $h$  is an integer and

$$h \leq \frac{m-1}{q} < h+1,$$

or

$$qh \leq m-1 < q(h+1).$$

Now because  $q$ ,  $h$  and  $m$  are *integers* we have

$$qh \leq m-1 \quad \text{if and only if} \quad qh < m$$

and

$$m-1 < q(h+1) \quad \text{if and only if} \quad m \leq q(h+1).$$

Thus  $h$  satisfies

$$qh < m \leq q(h+1)$$

or

$$h < \frac{m}{q} \leq h+1.$$

But this just means that  $h+1 = \left\lceil \frac{m}{q} \right\rceil$ . □

Note that the argument above depends very much on the fact that we are dealing with integers. The property we use is if  $m$  and  $n$  are any integers then  $m < n$  if and only if  $m + 1 \leq n$ .

The formula for  $K$  now becomes

$$K = \left\lfloor \frac{n}{q} \right\rfloor - \left\lfloor \frac{m-1}{q} \right\rfloor$$

or, if you prefer the notation used by many mathematicians,

$$K = \left[ \frac{n}{q} \right] - \left[ \frac{m-1}{q} \right].$$

In practice formulae such as the one for  $K$  are not very useful. You are almost always better off arguing directly. For example suppose you want to know how many multiples of 3 lie between 71 and 243. First you note that the first multiple of 3 in this range is 72, which is 24 times 3. The last multiple of 3 in the given range is 243 which happens to be 81 times 3. Thus the multiples of 3 which are of interest are  $24 \times 3, 25 \times 3, \dots, 81 \times 3$ . These multiples of 3 are indexed by the integers 24, 25,  $\dots$ , 81. Thus there are  $81 - 24 + 1 = 58$  of them. This approach has the advantage of generalizing easily to more complicated counting problems.

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