Multiple Equilibria in Heterogeneous Expectations Models

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Abstract

This paper demonstrates that heterogeneity in expectations may alter a model's regions of determinacy. We show how to associate with a heterogeneous expectations model (HE-model) a rational expectations model (ARE-model), the solutions to which are the equilibria of the HE-model. This association recasts the analysis of determinacy in the HE-model to analysis of determinacy in the ARE-model. We proceed to study a simple forward looking model and find that the stability properties of rational expectations models may not be robust. In particular, we present new results showing that in some models even a very small fraction of non-rational agents may preclude rational agents from coordinating on sunspots.

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1 Introduction

It is well-known that forward-looking linear rational expectations models may exhibit sunspot equilibria, and moreover, these sunspot equilibria may be stable under learning (Evans and Honkapohja 2002). These issues are not just theoretical curiosa. A large literature has developed studying indeterminacy in RBC-type models (Farmer 1999), and recently there has been considerable interest in whether monetary policy can rule out the existence of sunspot equilibria.[1] This paper addresses whether the existence of sunspot equilibria in rational expectations models is robust to heterogeneity in expectations.

[1]See (Woodford 2002) for an extensive overview.

Most research involving dynamic macroeconomic models imposes rational expectations and thus the assumption that agents' expectations are homogeneous. However, recent empirical evidence, indicates that expectations are heterogeneous. (Branch 2003) and (Carroll 2003) find that survey data suggest agents have a variety of methods which they use to form beliefs about future inflation. Heterogeneity across demographic groups was noted by (Bryan and Venkatu 2001a,b). In addition, recent theoretical work demonstrates that heterogeneity may arise in models where agents weigh the benefits and costs of various predictors. (Brock and Hommes 1997) show that when agents choose between costly rational and costless naive expectations agents will be distributed across both predictors across all time periods. In (Branch and Evans 2003) an equilibrium with heterogeneity is derived in which agents behave optimally in selecting between misspecified models of the economy.

Because the evidence for heterogeneous expectations is so compelling, we address its dynamic effects on an economy. We are primarily interested in the impact of heterogeneity on the existence of sunspot equilibria. We consider this issue in the context of a linear stochastic self-referential model. We define a *Heterogeneous Expectations Equilibrium* (HEE) as a stationary solution to the expectational difference equation given that expectations are formed heterogeneously. We then show how to write a heterogeneous expectations model (HE-model) as an associated rational expectations model (ARE-model), the solutions to which are also solutions to the HE-model. Thus we are able to test for the existence of sunspot equilibria in the usual way: by analyzing whether the ARE-model is determinate or indeterminate.

We find that imposing heterogeneous expectations may alter the regions of the parameter space corresponding to determinacy and indeterminacy. We illustrate this point in the context of the simplest heterogeneous expectations model; we assume agents are divided between rational and adaptive expectations. The determinacy of the HE-model depends on the weight assigned to each predictor and the weight assigned to past information in the adaptive predictor. When the adaptive predictor is naive (i.e. only looks at the most recent realization) and there is a unique REE then there will be a unique HEE. When there are multiple REE there will also be multiple HEE only if a majority of agents are rational, otherwise the model is explosive. We show if adaptive agents are mean-reverting then it may be possible for there to be a unique equilibrium in the HE-model even though under rational expectations sunspot equilibria exist; thus heterogeneity may have a stabilizing influence.[2] If adaptive agents are trend-setting then the HE-model may exhibit sunspot equilibria even though under rational expectations the equilibrium will be unique; thus heterogeneity may be destabilizing.

[2]Economies exhibiting sunspot equilibria are often described as unstable because agents may coordinate on one of many possible equilibrium paths.

2 A Theory of Heterogeneous Expectations Equilibria

2.1 REE and Indeterminacy

We consider reduced form models with the following recursive expectational structure:

$$z_t = \hat{A}\hat{E}_t z_{t+1} + \lambda' x_t \tag{1}$$

where $z_t \in \mathbb{R}$, $\hat{E}_t$ are (possibly non-rational) expectations formed at time t and x_t is a vector of exogeneous variables.[3] Forward looking models of this form arise in dynamic stochastic general equilibruim models, as well as in ad hoc models such as the Cagan model of inflation. For simplicity assume that x_t is zero mean iid.

A rational expectations equilibrium (REE) of the reduced form model is a (weakly) stationary solution to the expectational difference equation (1) given $\hat{E} = E$. One such solution is $z_t = \lambda' x_t$. It can also be easily verified that, provided $|A| > 1$, there are solutions of the form

$$z_{t+1} = A^{-1} z_t - A^{-1}\lambda' x_t + \epsilon_{t+1} \tag{2}$$

where ϵ_t is a martingale difference sequence. Notice that, in this case, multiple equilibria exist. Solutions of this form are often referred to as sunspot solutions since they depend on extrinsic noise.

There is an established literature that treats the existence of multiple equilibria in rational expectations models: see (Blanchard and Khan, 1980), (Farmer, 1999), and (Evans and Honkapohja, 2001) for details. This literature provides the following terminology: A (linear) model is said to be *determinate* if it has a unique non-explosive equilibrium path and *indeterminate* if there are multiple such paths. Whether a model is determinate depends on the number of explosive forward roots relative to the number of free variables in the model; again, details may be found in the above citations.

2.2 Heterogeneous Expectations

We now relax the assumption that $\hat{E} = E$. For expositional ease, we assume that $\hat{E}$ represents a convex combination rational and adaptive agents, that is,

$$\hat{E}_t(y_{t+1}) = \alpha E_t y_{t+1} + (1-\alpha) E_t^* y_{t+1} \tag{3}$$

[3] We consider the univariate case for simplicity, but there is nothing in our development which prevents considering multi-variate models.

where

$$E_t^* y_{t+1} = \gamma_0 + \sum_{k=1}^{p} \gamma_k y_{t-k}. \tag{4}$$

A *heterogenous expectations equilibrium* (HEE) is any stationary solution to the expectational difference equation. Our approach also applies to other expectation operators. We make the above restriction to remain close to the adaptive learning literature and to focus only on expectations derived from commonly used forecasting methods.[4]

In this paper we are primarily interested in comparing the number of equilibrium sequences under HE to the number of equilibrium sequences under RE, and so we extend the notion of uniqueness under RE to the HE model. An HE-model is said to be determinate if there exists a unique stationary solution to (1) given (3), (4). To analyze the determinacy of an HE model, we write it in terms of an associated RE-model (ARE); equation (1) becomes

$$z_t = \beta E_t z_{t+1} + \delta \left(\gamma_0 + \sum_{k=1}^{p} \gamma_k y_{t-k} \right) + \lambda' x_t, \tag{5}$$

where $\beta = \alpha \hat{A}$ and $\delta = \hat{A}(1-\alpha)$. Notice that not only is the coefficient modifying the conditional expectations operator altered (as compared to the original model under rational expectations), but also p lagged, and hence predetermined, variables are introduced.[5] The insight of this approach is solutions to (5), the ARE-model, are also solutions to (1), the HE-model.

[4] An important and slippery issue is whether it is theoretically appropriate to replace rational expectations with heterogeneous expectations at the reduced form level, that is, after aggregation across the economy has already occurred. In particular, many reduced form models are derived under the assumption of a representative agent, where as, in case of heteregeneous expectations, a representative agent does not exist. For simplicity, we sidestep this issue here, but note that, provided different agent types are forecasting the same aggregate variable (e.g. inflation and not individual consumption level) then replacement of expectations operators at the reduced form level can be justified.

[5] If the model is taken to be doubly infinite, then the introduction of predetermined variables is not particularly troubling. However, if the model is taken to begin at some time t_0 then the predetermined variables must have initial conditions. It is natural, and not difficult to show formally, that choosing the initial conditions is equivalent to choosing non-rational agents' initial expectations. Since we have not modelled the initial expectations choices of non-rational agents, we will assume that these initial conditions are chosen randomly, and, in particular, do not necessarily satisfy conditions required for non-explosive behavior, such as fixing the economy to lie in a stable manifold.

3 Altering Determinacy

We are now in a position to examine the implications of introducing heterogeneity into a linear model. Consider the expectational difference equation

$$y_t = \beta \hat{E}_t y_{t+1}. \tag{6}$$

Notice that we have shut down the stochastic term. The results in this section extend to the stochastic model as well; we make this assumption to simplify the analysis. We restrict attention in this section to $\beta > 0$. If $\beta < 1$ the unique non-explosive REE is $y_t = 0$. When $\beta > 1$ then the model with all agents acting rationally (RE-model) also admits so-called sunspot solutions,

$$y_t = \beta^{-1} y_{t-1} + \varepsilon_t,$$

where ε_t is a martingale difference sequence. Thus, the model is indeterminate if and only if $\beta > 1$. This result is not new. The question we now pose is, "Does heterogeneity alter the region of indeterminacy?"

To answer this question it suffices to analyze the simple case in which a proportion α of agents are rational and the rest believe in mean reversion:

$$\hat{E}_t y_{t+1} = \alpha E_t y_{t+1} + (1-\alpha)\theta y_{t-1}. \tag{7}$$

The associated rational model is now

$$y_t = \gamma E_t y_{t+1} + \delta y_{t-1}, \tag{8}$$

where $\gamma = \beta\alpha$ and $\delta = \beta\theta(1-\alpha)$. This model has one predetermined variable (y_{t-1}) and one free variable (y_t). Any solution to (8) must satisfy

$$y_t = \gamma^{-1} y_{t-1} - \gamma^{-1}\delta y_{t-2} + \varepsilon_t$$

where ε_t is a martingale difference sequence. Whether this ARE-model is determinate, indeterminate, or explosive depends on the parameters γ and δ as noted in the Appendix.

In case of naive agents ($\theta = 1$), we have the following proposition:

Proposition 1 *If $\theta = 1$ the following hold.*

1. *If $\alpha > 1/2$ then indeterminacy in the RE-Model (6) implies the HE-model (6) with expectations given by (7) is indeterminate.*

2. *If $\alpha < 1/2$, indeterminacy in the RE-model implies the HE-model is explosive.*

3. *If the RE-model is determinate then the HE-model is determinate for all* $0 \leq \alpha \leq 1$.

The proofs of all propositions may be found in the Appendix.

We conclude that this type of heterogeneity does not alter the region of indeterminacy if the proportion of naive agents is not too large. However, when the naive agents form a majority then the HE-Model may be explosive even though the RE-Model is indeterminate. This result signals a cautionary note to modelers: mistakenly assuming rational expectations in a model may result in modeling choices which bring about a model with no stationary equilibria. When considering existence of equilibria in a rational expectations model the existence in a heterogeneous expectations model should also be considered. In our simple example, the result depends on the proportion of naive. However, recent empirical evidence suggests that the proportion of adaptive agents may be high.[6]

Suppose now that $0 < \theta < 1$. We have the following proposition:

Proposition 2 *Let* $0 < \theta < 1$.

1. *If the RE-model is determinate then the HE-model is determinate.*

2. *If* $\theta < (1 - \alpha\beta) / (\beta(1 - \alpha))$ *then indeterminacy in the RE-model implies determinacy in the HE-model.*

Proposition 2 provides parameter values for which the RE-model is indeterminate and the associated HE-model is determinate. We conclude that heterogeneity of this type may have a stabilizing effect on the economy (here stabilizing is in the sense that there is a unique HEE).[7] And for some model parameterizations, only a small fraction of adaptive agents are required; in particular, as $\beta \to 1$, the required proportion of adaptors goes to zero.

The results of proposition 2 are significant. A sizeable literature considers the effects of small deviations from rationality.[8] We are unaware of any paper, however, which examines the effect of departures from rationality on the existence of multiple equilibrium sequences. Proposition 2 presents a new result. Models which have sunspot equilibria exhibit an infinite number of possible sequences on which agents

[6] See, for example, (Branch 2003) and (Carroll 2003).

[7] This so-called saddle path stability is frequently argued as an important criteria for rational expectations models because it rules out bubble-solutions and makes coordination on equilibria easier. For a discussion of the debate over selection criteria for multiple equilibria models see (McCallum 1999). For a discussion of the connection between determinacy and stability of learning see (Bullard and Mitra 2001). A discussion of determinacy, coordination on sunspot equilibria, and learnability is addressed in (Gauthier 2000, 2002).

[8] See, for example, (Anderlini and Canning 2001).

may coordinate. Under some conditions, a small number of adaptive agents may preclude such coordination. This indicates that sunspot equilibria are less robust than previously thought.

Finally, suppose $\theta > 1$; we interpret this somewhat unlikely operator as suggesting that a proportion of agents have explosive expectations. Although unlikely, $\theta > 1$ is not unreasonable. It is considered, for instance, in (Pesaran 1987). If agents have trend-expectations so that they expect past trends to accelerate then this parameter restriction will fit. Trend followers, for example, seem to fit such a scheme. We have the following result:

Proposition 3 *Let $\theta > 1$. If the RE-model is indeterminate then*

1. *if $\alpha > \theta / (1 + \theta)$ the HE-model is indeterminate.*
2. *if $\alpha < \theta / (1 + \theta)$ the HE-model is explosive.*

If the RE-model is determinate then if $\alpha/(1-\alpha) > \theta$ and $1/(\theta(1-\alpha)+\alpha) < \beta$ the HE-model is indeterminate.

For the case of a proportion of trend followers, the original RE-model may be determinate and the HE-model indeterminate. We conclude that heterogeneity of this type may have a de-stabilizing effect on the economy. Moreover, there exist equilibria which depend on extrinsic noise (i.e. sunspots) in the heterogeneous expectations model when the RE-model does not have any such equilibria. This result is interesting and suggests that if some agents are trend-followers it is possible for rational agents to coordinate on 'sunspots' when they would not under full-rationality.

4 Conclusion

The results of this paper show that in a simple, linear, non-stochastic model, the number and nature of equilibria are highly dependant on the specification of agents' expectations. If expectations are heterogeneous then for certain parameterizations there is a correspondence between determinacy and indeterminacy in RE-models and HE-models. However, for other parameterizations the HE-model stabilizes an otherwise unstable RE-model. And finally, if expectations follow a trend-setting scheme then the economy may have sunspot equilibria under heterogeneous expectations when it would not under rational expectations.

Our results suggest there is no generic relationship between the number of equilibria in a model under rational expectations and the same model under heterogeneous expectations. The inclusion of heterogeneity may have a stabilizing or destabilizing effect, or even preclude the existence of equilibria. We have presented a method for analyzing the effects of heterogeneity, and these results strongly suggest this analysis is important when working with forward looking models. Incorrectly assuming rational expectations may lead the modeler to predict outcomes far different from what the model actually exhibits.

The results here are particularly important for models in which the determinacy of RE-models depends on policy parameters. For example, in monetary models such as (Woodford 2002) it is often suggested that monetary policy should follow an active Taylor rule in order to rule out multiple rational expectations equilibria. Our results indicate that if policymakers unwittingly assume agents have rational expectations they may destabilize an already stable system. In future work we will examine this very issue in order to specify robustness criteria for monetary policy rules.

Appendix

The following result is well-known, and can be found, for example, in (Evans and Honkapohja, 2001).

Lemma 1 *The heterogeneous expectations model (1) with the expectations operator (7) model is indeterminate if and only if the pair* (γ, δ) *satisfy either of the two conditions*

1. $\gamma > 1/2$, $\delta > 1 - \gamma$, *and* $\delta < \gamma$.
2. $\gamma < -1/2$, $\delta < -1 - \gamma$, *and* $\delta > \gamma$.

The model is determinate if and only if (γ, δ) *satisfy both of the conditions*

1. $\delta < 1 - \gamma$
2. $\delta > -\gamma - 1$

Proof of Proposition 1 The proof of this result and the others relies on the observation that $\delta = \theta(\beta - \gamma)$. Assume $\theta = 1$. To prove part 1, assume $\beta > 1$ so that the RE-model is indeterminate. Then $\alpha > 1/2$ implies $\gamma = \beta\alpha > 1/2$. Also, $\alpha > 1/2 \Rightarrow \gamma = \beta\alpha > \beta(1 - \alpha)$. Finally, $\beta > 1 \Rightarrow \delta = \beta - \gamma > 1 - \gamma$, so that the HE-model is indeterminate by Lemma 4. To prove part 2 we must show explosiveness, which, since $\beta > 0$, requires $\delta > \gamma$ and $\delta > 1 - \gamma$. Thus let $\beta > 1$ and $\alpha < 1/2$. Then $\beta > 1 \Rightarrow \delta = \beta - \gamma > 1 - \gamma$ and $\alpha < 1/2 \Rightarrow \delta = \beta(1 - \alpha) > \beta/2 > \gamma$. Finally, to prove part 3, let $\beta < 1$. To show determinacy, we must show $\delta, \gamma > 0$ and $\delta < 1 - \gamma$. But δ and γ are positive by construction and $\beta < 1 \Rightarrow \delta = \beta - \gamma < 1 - \gamma$. ■

Proof of Proposition 2 Let $\theta < 1$, $\beta < 1$. Again δ and γ are positive by construction, so we must show that $\delta < 1 - \gamma$. But $\delta = \theta(\beta - \gamma) < \theta(1 - \gamma) < 1 - \gamma$. This proves part 1. Part 2. is simply algebra. ■

Proof of Proposition 3 Let $\beta > 1$ and $\theta > 1$, and assume $\alpha > \theta/(1 + \theta)$. First notice $\delta = \theta(\beta - \gamma) > \beta - \gamma > 1 - \gamma$. Also, $\gamma > \beta\theta/(1 + \theta)$ so that $\gamma(1 + \theta) > \theta\beta$ or $\gamma > \theta(\beta - \gamma) = \delta$. Finally, $1 - \gamma < \delta < \gamma \Rightarrow \gamma > 1/2$. This proves part 1. To prove part 2, notice that $\delta = \theta(\beta - \gamma) > \beta - \gamma > 1 - \gamma$ still holds. Also, if $\alpha < \theta/(1 + \theta)$, the same argument as above with the inequalities reversed shows that $\gamma < \beta\theta/(1+\theta)$ implies $\gamma < \delta$. ■

References

[1] Anderlini, Luca, and David Canning, 2001, "Structural Stability Implies Robustness to Bounded Rationality," *Journal of Economic Theory*, 101, 395-422.

[2] Blanchard, Olivier Jean, and Charles M. Khan, 1980, "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48, 5, 1305-1311.

[3] Branch, William A., 2003, "The Theory of Rationally Heterogeneous Expectations: Evidence from Survey Data on Inflation Expectations," mimeo.

[4] Branch, William A., and George W. Evans, 2003, "Intrinsic Heterogeneity in Expectation Formation," mimeo. 129-1160.

[5] Brock, William A., and Cars H. Hommes, 1997, "A Rational Route to Randomness," *Econometrica*, 65, 1059-1160.

[6] Bryan, Michael F., and Guhan Venkatu, 2001a, "The Demographics of Inflation Opinion Surveys," *Economic Commentary*, Federal Reserve Bank of Cleveland.

[7] Bryan, Michael F., and Guhan Venkatu, 2001b, "The Curiously Different Inflation Perspectives of Men and Women," *Economic Commentary*, Federal Reserve Bank of Cleveland.

[8] Bullard, James M., and Kaushik Mitra, 2002, "Learning about Monetary Policy Rules," *Journal of Monetary Economics*, 49, 1105-1129.

[9] Carroll, Christopher D., 2003, "The Epidemiology of Macroeconomic Expectations," *Quarterly Journal of Economics*, 118,1.

[10] Evans, George W., and Seppo Honkapohja, 2001, *Learning and Expectations in Macroeconomics*, Princeton University Press.

[11] Evans, George W., and Seppo Honkapohja, 2002, "Expectational Stability of Stationary Sunspot Equilibria in a Forward-looking Linear Model," forthcoming *Journal of Economic Dynamics and Control*.

[12] Farmer, Roger, 1999, *Macroeconomics of Self-Fulfilling Prophecies,* MIT Press.

[13] Gauthier, Stephane, 2001, "Determinacy and Stability under Learning of Rational Expectations Equilibria," *Journal of Economic Theory*, 102, 354-374.

[14] Gauthier, Stephane, 2002, "Small Departures from Rationality Magnify Fluctuations," mimeo.

[15] McCallum, Bennett T., 1999, "Issues in the Design of Monetary Policy Rules," *Handbook of Macroeconomics*, Vol 1c, ed. by J.B. Taylor and M. Woodford. North-Holland, 1999.

[16] Pesaran, M. Hashem, 1987, *The Limits to Rational Expectations,* Basil Blackwell.

[17] Woodford, Michael, 2002, *Interest and Prices*, manuscript.