

Stable Sunspot Solutions in Models with Predetermined Variables*

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Abstract

We consider a linear stochastic univariate rational expectations model, with a predetermined variable, and provide alternative representations of SSEs (stationary sunspot equilibria). For a strict subset of the parameter space there exist SSEs that are locally stable under least squares learning provided agents use a common factor representation for their estimated law of motion. These results indicate that for some parameter regions SSEs are more likely to arise under private agent learning than previously recognized.

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1 Introduction

The possibility of expectations driven fluctuations in economic activity, emphasized by (Keynes 1936), received new support with the work of (Shell 1977), (Azariadis 1981), (Cass and Shell 1983) and (Guesnerie 1986), in

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which it was shown that some market clearing models could have “self-fulfilling” solutions, under rational expectations, driven by extraneous stochastic processes known as “sunspots.” Although sunspot equilibria were first shown to exist in simple stylized models, such as the Overlapping Generations model of money¹, the line of research initiated by (Benhabib and Farmer 1994), (Farmer and Guo 1994) and (Farmer 1999) has shown the possibility of sunspot equilibria in versions of Real Business Cycle models with monopolistic competition or externalities.

A question that has accompanied the growth of this literature concerns the attainability of such equilibria. That sunspot solutions could be stable under adaptive learning was demonstrated for the basic Overlapping Generations model by (Woodford 1990), and conditions for local stability under adaptive learning were provided in (Evans and Honkapohja 1994b) and (Evans and Honkapohja 2002a) for one-step forward looking univariate nonlinear models. The solutions considered in these papers take the form of a dependence on extraneous finite state Markov processes. This type of solution is prominent in the theoretical literature, e.g. see (Chiappori and Guesnerie 1989), but plays a relatively minor role in applied macroeconomic research.

A different class of sunspot solutions, taking an autoregressive-moving average (ARMA) form, has also been discussed in the literature, e.g. (McCallum 1983), (Farmer 1999), and Part III of (Evans and Honkapohja 2001). In these formulations the extraneous sunspot variable is an exogenous white noise process. (Evans and Honkapohja 1994a) and (Evans and Honkapohja 2001) discuss the stability under learning, for linear models with predetermined variables, of sunspot solutions taking this ARMA form, which is the type of solution more commonly considered in the business cycle literature.

The current paper reexamines these issues in the context of a linear expectations model with a predetermined variable. Our major contribution is to show that a subset of ARMA-type stationary sunspot equilibria, which would previously have been viewed as unstable under learning, are stable when agents condition their forecasts on an appropriate exogenous variable. This finding substantially increases the set of stationary sunspot equilibria that are plausible solutions, in the sense of being locally learnable, and these include the types of equilibria often emphasized in applied work.

We stress throughout this paper that rational expectations equilibria have

¹See the extensive survey in (Guesnerie and Woodford 1992).

alternative representations, and we show how stability under learning depends in part on the representation estimated by agents and used by them to make forecasts. Although the “general form” ARMA representations that depend on serially uncorrelated sunspot variables, are not stable under learning, we find that alternative “common factor” representations of these solutions are in some cases stable. One way to view our results is that they show how stability under learning of sunspot equilibria can depend on the time-series structure of the observed sunspot variable. For part of the parameter space, exogenous autoregressive variables with an appropriate autocorrelation coefficient (a “resonant frequency”) can induce sunspot equilibria that are stable under learning.

2 The Framework

We consider the following univariate model:

$$y_t = \alpha + \beta E_t^* y_{t+1} + \delta y_{t-1} + v_t,$$

where v_t is a white noise exogenous process and $E_t^* y_{t+1}$ denotes the expectations formed by agents at the start of period t . Under the rational expectations assumption $E_t^* y_{t+1} = E_t y_{t+1}$, where $E_t y_{t+1}$ is the mathematical expectation of y_{t+1} conditional on information available at t .

Clearly we want to impose that $\beta \neq 0$ and $\delta \neq 0$ and we will also assume that $\beta + \delta \neq 1$. There is then a unique nonstochastic steady state, when $v_t \equiv 0$, given by $\bar{y} = (1 - \beta - \delta)^{-1} \alpha$. Hereafter we let $y_t, y_{t-1}, E_t^* y_{t+1}$, etc., denote deviations from \bar{y} , and we can therefore rewrite the model as²

$$y_t = \beta E_t^* y_{t+1} + \delta y_{t-1} + v_t. \tag{1}$$

Solutions to (1) will depend on v_t , and may also depend on other extraneous stochastic processes, usually called “sunspots.” In this paper we begin by characterizing the rational expectations solutions to (1). We then relax the assumption that expectations are given by true conditional expectations and consider the model under learning.

A *rational expectations equilibrium* (REE) is a stochastic process y_t that satisfies (1) under the assumption that $E_t^* y_{t+1} = E_t y_{t+1}$. The model (1) can

²The set of solutions to the model without predetermined variables was discussed in detail by (Gourieroux, Laffont, and Monfort 1982).

be regarded as holding for all integers t , in which case REE solutions y_t are doubly infinite stochastic processes. Alternatively the model can be assumed to hold for all integers $t \geq t_0$, where without loss of generality we set $t_0 = 0$. In the latter case REE are solutions that begin at time $t = 0$ with some initial distribution y_0 , which for convenience we take to be a nonstochastic initial condition \check{y}_0 . In this case we say that the REE y_t is an initialized stochastic process. We will focus on the doubly infinite case, but occasionally will need to explicitly discuss initialized models.

We restrict attention to solutions that are nonexplosive in the sense that, for each t , $E_t |y_{t+s}|$ is uniformly bounded over s . The applied literature frequently focuses on solutions that are covariance stationary, in the case of doubly infinite processes, or asymptotically covariance stationary, in the case of initialized processes. It can be shown that if a process is covariance stationary, or asymptotically covariance stationary, then it is uniformly bounded in expectation, and that the latter condition in turn implies that it is conditionally uniformly bounded in expectation. A statement and proof of these results is provided in the Appendix.

Whether (1) has a nonexplosive solution depends on the roots of the associated quadratic $\beta a^2 - a + \delta$,

$$a_1 = \frac{1 - \sqrt{1 - 4\beta\delta}}{2\beta} \text{ and } a_2 = \frac{1 + \sqrt{1 - 4\beta\delta}}{2\beta}.$$

Throughout the paper, for convenience, we will assume that the two roots are distinct and that neither lies on the unit circle. If both roots are greater than one in norm³ then, as we show below, there are no nonexplosive solutions to the model. If the quadratic $\beta a^2 - a + \delta$ has exactly one real root a_i with norm less than one then there is a stationary solution given by

$$z_t^i = (1 - a_i L)^{-1} (1 - \beta a_i)^{-1} v_t = (1 - \beta a_i)^{-1} \sum_{j=0}^{\infty} a_i^j v_{t-j}. \quad (2)$$

Here L is the lag operator defined by $Ly_t = y_{t-1}$ and (2) explicitly gives the (absolutely summable) moving average representation of the solution z_t^i . It is easily verified that z_t^i solves (1) and below we show that it is the unique nonexplosive solution in this case. This solution, of course, has an AR(1),

³i.e., have absolute value greater than one if real or complex modulus greater than one if not real.

i.e. first order autoregressive, representation, which can be written as

$$\begin{aligned} z_t^i &= a_i z_{t-1}^i + (1 - \beta a_i)^{-1} v_t, \text{ or equivalently} \\ z_t^i &= a_i z_{t-1}^i + (\beta a_j)^{-1} v_t, \text{ for } j \neq i, \end{aligned} \quad (3)$$

where the second line uses $\beta(a_1 + a_2) = 1$.

If both roots of the associated quadratic are less than one in norm then there is a stationary solution given by

$$\hat{z}_t = (1 - \beta^{-1}L + \beta^{-1}\delta L^2)^{-1}(1 - \beta^{-1}L)v_t,$$

as can again be directly verified. Note that this can also be written as the ARMA(2,1) process $\hat{z}_t = \beta^{-1}\hat{z}_{t-1} - \beta^{-1}\delta\hat{z}_{t-2} + v_t - \beta^{-1}v_{t-1}$. If the roots are real there also exist two solutions of the form (2), or equivalently (3), in which a_i is chosen to be either of the two roots. These solutions are often referred to as the MSV (minimal state variable) solutions, following (McCallum 1983).⁴ However, if both roots are less than one in norm there are other stationary solutions as well, and the complete class of solutions can be described in various ways. To fix ideas, we provide a characterization of the full set of solutions in terms of martingale difference sequences. Although these and related results on regularity and determinacy are not new, they provide the starting point for our analysis of the stability of solutions under learning.

Let $\varepsilon_{t+1} = y_{t+1} - E_t y_{t+1}$ be agents' forecast error, and note that rational expectations implies $E_t \varepsilon_{t+1} = 0$. A stochastic process ε_t that satisfies this property is known as a martingale difference sequence (mds). It follows that a stochastic process y_t that solves (1) must also satisfy

$$y_{t+1} = \beta^{-1}y_t - \beta^{-1}\delta y_{t-1} - \beta^{-1}v_t + \varepsilon_{t+1} \quad (4)$$

for some martingale difference sequence ε_{t+1} . Conversely, for any arbitrary martingale difference sequence ε_{t+1} this equation specifies a solution, and thus an equilibrium provided the solution is not explosive.⁵

⁴In fact (McCallum 1983) recommends choosing one of these two solutions based on a subsidiary selection principle, whereas we will focus on local stability under learning of alternative solutions.

⁵The approach of writing solutions to linear rational expectations models in terms of arbitrary martingale difference sequences was emphasized by (Broze, Gourieroux, and Szafarz 1990).

The concept of “regularity” will help us characterize the restrictions necessary for non-explosive solutions to obtain. Begin by rewriting the potential solution (4) in matrix form:

$$\begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix} = \begin{bmatrix} \beta^{-1} & -\beta^{-1}\delta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} -\beta^{-1} \\ 0 \end{bmatrix} v_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \varepsilon_{t+1}. \quad (5)$$

Let A be the coefficient matrix of the lagged state vector in (5). If the number of roots of A with norm less than one is equal to (more than, less than) the number of predetermined variables then the model is said to be regular (irregular, explosive). The characteristic values of A are the roots of quadratic $\beta a^2 - a + \delta$, and in the case at hand there is one predetermined variable. Thus, as suggested by the discussion above, we will find that if the model is regular then there is a unique REE and thus a unique mds ε_{t+1} for which (4) is stationary; and, if the model is irregular then there are multiple REE and (4) is stationary for any mds ε_{t+1} . Note that when the model is irregular, there are two equilibria corresponding to MSV solutions. We will call equilibria that are not MSV solutions *stationary sunspot equilibria*, or SSEs, and the associated martingale differences sequences ε_t will be called *sunspots*.⁶

A closely related concept is that of “determinacy”. This concept is most naturally defined in terms of initialized equilibrium processes. Let $\{y_t\}_{t=0}^{\infty}$ be a solution to (1) subject to the initial condition $y_0 = \check{y}_0$. The solution is said to be indeterminate if there exists a distinct solution $\{y'_t\}_{t=0}^{\infty}$, meeting the initial condition, such that $\lim_{t \rightarrow \infty} |y_t - y'_t| = 0$ and $\sup_{t \geq 1} |y_t - y'_t|$ can be made arbitrarily small, where equality here is taken to hold almost everywhere. Otherwise the solution is said to be determinate.

We have the following result:

Proposition 1 *Let a_1, a_2 be the roots of the associated quadratic. (i) If $|a_1|, |a_2| > 1$ then almost surely there exist no nonexplosive solutions to (1). (ii) In the regular case $|a_1| < 1 < |a_2|$ there is a unique nonexplosive solution to (1). This solution has the representation*

$$y_t = a_1 y_{t-1} + (\beta a_2)^{-1} v_t,$$

which is stationary, in the doubly infinite case, and asymptotically stationary for the initialized model. It follows that there do not exist SSEs. (iii) In the

⁶(Blanchard and Kahn 1980) showed, for a general class of linear models, that there is a unique nonexplosive solution in the “regular” case (see below).

irregular case $|a_1|, |a_2| < 1$ SSEs exist. y_t is a solution if and only if it can be written in the form (4) for some martingale difference sequence ε_{t+1} , and in this case every solution to the initialized model is indeterminate.

The proof is in the Appendix. The region of the parameter space in which SSEs exist is given by the following:

Corollary 2 *The model is irregular if and only if the parameters of the model are contained in one of the following two regions:*

1. $\beta > \frac{1}{2}, \quad \delta > 1 - \beta, \quad \delta < \beta$
2. $\beta < -\frac{1}{2}, \quad \delta < -1 - \beta, \quad \delta > \beta$

This result follows from the well-known requirement for irregularity (in a two dimensional environment) that $|\det A| < 1$ and $|\text{tr}(A)| < 1 + \det(A)$, where tr and \det denote trace and determinant, respectively. It is also straightforward to compute that the regular case corresponds to the region $|\beta + \delta| < 1$.

3 Representations and T-maps

Before considering the stability of REE under learning we must first distinguish between REEs and their representations. We begin by defining the notion of representations and then show how maps from the perceived to the actual law of motion, which we will call T-maps, can be used to characterize them. As will be shown later, alternative representations of an REE correspond to alternative econometric specifications that might naturally be used by an economic agent faced with the problem of how to forecast key variables required for decision making.

3.1 Representations

Recall the model (1) under the RE assumption:

$$y_t = \beta E_t y_{t+1} + \delta y_{t-1} + v_t,$$

reproduced here for convenience. Recall that an REE is a stochastic process y_t that satisfies this equation. A *rational expectation equilibrium representation* (REER) of a given REE is a linear recursion, a solution to which

is the REE. We say the REE is represented by the REER. Notice that a given REE may have more than one representation: for example if a doubly infinite REE is represented by an AR(1) process then it is also represented by an ARMA(2,1); simply operate on each side of the AR(1) representation by $1 - \eta L$ for any $\eta \in \mathbb{R}$. Also, a given REER may have many solutions and hence yield many REE's: for example, if an initialized model does not specify a full complement of initial conditions for the REER then the REER may have infinitely many solutions corresponding to the additional degrees of freedom in its initial conditions.

In the previous section we showed that, when the nonexplosiveness condition is imposed, in the regular case there is a unique REE. In this case the REE does not depend on sunspots and it has an AR(1) representation driven by the intrinsic shock. In contrast, in the irregular case there are multiple REE, including REE that depend on sunspots. We now consider in more detail representations of the REE in the irregular case. In the previous section we demonstrated the first result on representations of REEs, namely:

Proposition 3 *In the irregular case, the process y_t is an REE of (1) if and only if there is a martingale difference sequence ε_t such that y_t solves*

$$y_t = \beta^{-1}y_{t-1} - \beta^{-1}\delta y_{t-2} - \beta^{-1}v_{t-1} + \varepsilon_t. \quad (6)$$

Representations of this form we call *general form representations*.

If we consider only the case with real roots and doubly infinite processes, we also have the following:

Proposition 4 *Consider the irregular case and assume that the roots of the associated quadratic are real. The process y_t is an REE of (1) if and only if there is a martingale difference sequence ε_t such that y_t solves*

$$y_t = a_i y_{t-1} + \xi_t + (\beta a_j)^{-1} v_t \quad (7)$$

where a_i , $i = 1, 2$ are the roots of the quadratic $\beta a^2 - a + \delta = 0$, $i \neq j$, and ξ_t satisfies

$$\xi_t = a_j \xi_{t-1} - (\beta a_j)^{-1} v_t + \varepsilon_t. \quad (8)$$

Proof. By Proposition 3 y_t is an REE if and only if there exists a martingale difference sequence. ε_t such that y_t solves (6). Define the process

ξ_t by (8). Then y_t is an REE if and only if

$$\begin{aligned} (1 - \beta^{-1}L + \beta^{-1}\delta L^2)y_t &= \varepsilon_t - \beta^{-1}v_{t-1} \\ \Leftrightarrow (1 - a_1L)(1 - a_2L)y_t &= \xi_t - a_j\xi_{t-1} - \beta^{-1}v_{t-1} + (\beta a_j)^{-1}v_t \\ \Leftrightarrow (1 - a_1L)(1 - a_2L)y_t &= (1 - a_jL)\xi_t + (\beta a_j)^{-1}(1 - a_jL)v_t. \end{aligned}$$

Canceling the appropriate lag polynomial yields the result. ■

We will refer to representations of this form as *common factor representations* (CF representations) since they are obtained from lag polynomials with a common factor.⁷ These propositions tell us that each doubly infinite REE can be represented in general form and (in two ways) as a common factor. Several additional remarks are in order.

Remark 1: We can equivalently state that y_t is an REE if and only if there is a martingale difference sequence $\tilde{\varepsilon}_t$ such that y_t solves (7) and ξ_t satisfies

$$\xi_t = a_j\xi_{t-1} + \tilde{\varepsilon}_t.$$

This is immediate since $\varepsilon_t = \tilde{\varepsilon}_t + (\beta a_j)^{-1}v_t$ defines a martingale difference sequence.

Remark 2: Each of the two MSV solutions $y_t = a_i y_{t-1} + (\beta a_j)^{-1}v_t$ also has two CF representations. One representation is $y_t = a_i y_{t-1} + \xi_t + (\beta a_j)^{-1}v_t$ with $\xi_t \equiv 0$, and the other representation is given by $y_t = a_j y_{t-1} + \xi_t + (\beta a_i)^{-1}v_t$ with $\xi_t = a_i \xi_{t-1} + \{(\beta a_j)^{-1} - (\beta a_i)^{-1}\}v_t$.

Remark 3: In the initialized case the analog of the CF representation of an REE y_t is

$$y_{t+1} = a_i y_t + \xi_{t+1} + (\beta a_j)^{-1}v_{t+1} + K a_j^{t+1},$$

for $t = 0, 1, 2, \dots$, and K is arbitrary. Since $|a_j| < 1$ as $t \rightarrow \infty$ the solution y_t converges to a process that satisfies (7).

3.2 T-maps

A useful way to analyze representations of REE's is to view them as fixed points of a map. Specifically, assume agents know the functional form of the representation, but do not know the corresponding parameter values; we

⁷The importance of common factor solutions, for REE driven by the fundamental shock v_t , was stressed in (Evans and Honkapohja 1986). For other applications see (Evans and Honkapohja 1994a) and (Evans and Honkapohja 2002b).

assume they have some perception of what the parameter values are and we call the subsequent functional form with these values imposed the *perceived law of motion* (PLM). Agents use the PLM to form their expectations of future values of y_t . The *actual law of motion* (ALM) is obtained by inserting these expectations into the reduced form model. Provided the PLM is well-specified, the ALM will have the same functional form and the associated parameter values, which we think of as the actual parameter values, will depend on the agents' perceptions. If θ represents the agents' perceived parameter values, let $T(\theta)$ represent the actual parameter values. Then a fixed point of this T – *map* determines an REER.

To analyze the REER's of our model, consider the case of real roots and let ξ_t be an observable stationary exogenous process defined by

$$\xi_t = \lambda \xi_{t-1} + \varepsilon_t,$$

where $E_t \varepsilon_{t+1} = 0$ and $|\lambda| < 1$. We assume that λ is known. Particular values of λ will be of interest, but we do not at this stage impose any additional restrictions on λ . In line with the preceding results we consider PLMs of the form

$$y_t = ay_{t-1} + by_{t-2} + c + d\xi_t + kv_t + lv_{t-1}. \quad (9)$$

In terms of the general notation above we are thus setting

$$\theta = (a, b, c, d, k, l).$$

Note that this specification for the PLM includes the general form solutions (provided $\lambda = 0$) and the common factor solutions (provided $\lambda = a_j$).

For this PLM expectations are formed according to

$$\begin{aligned} E_t^* y_{t+1} &= c + aE_t^* y_t + by_{t-1} + d\lambda \xi_t + lv_t \\ &= c + a(c + ay_{t-1} + by_{t-2} + d\xi_t + kv_t + lv_{t-1}) + by_{t-1} + d\lambda \xi_t + lv_t, \end{aligned}$$

so that

$$E_t^* y_{t+1} = c(1+a) + (a^2 + b)y_{t-1} + aby_{t-2} + d(a+\lambda)\xi_t + (l+ak)v_t + alv_{t-1}. \quad (10)$$

Note that here we make the assumption, frequently made in the literature, that y_t is not part of the time t information set, but that exogenous variables ξ_t and v_t and lagged variables are known when forecasts are made at t . (Section 6 discusses the impact of including y_t in the information set).

Substituting (10) into (1) yields the ALM

$$y_t = (\beta(a^2 + b) + \delta)y_{t-1} + \beta a b y_{t-2} + \beta(1+a)c + \beta d(a + \lambda)\xi_t \\ + (1 + \beta l + \beta a k)v_t + \beta a l v_{t-1},$$

and thus the T-map is given by

$$a \longrightarrow \beta(a^2 + b) + \delta, \quad (11)$$

$$b \longrightarrow \beta a b, \quad (12)$$

$$c \longrightarrow \beta(1+a)c, \quad (13)$$

$$d \longrightarrow \beta(a + \lambda)d, \quad (14)$$

$$k \longrightarrow (1 + \beta l + \beta a k), \quad (15)$$

$$l \longrightarrow \beta a l. \quad (16)$$

The functional form of the PLM is sufficiently general that the fixed points of this T-map will capture both the general form representations and the common factor representations. To analyze these fixed points it is helpful to begin by considering the subsystem of $T(\theta) = \theta$ corresponding to (11) and (12). Provided that the quadratic $a = \beta a^2 + \delta$ has distinct real roots a_1 and a_2 , there are three possible solutions for (a, b) , given by $(a_1, 0)$, $(a_2, 0)$ and $(\beta^{-1}, -\beta^{-1}\delta)$.

There are thus two cases: (I) If $(a, b) = (\beta^{-1}, -\beta^{-1}\delta)$ then it is easily seen that fixed points θ must satisfy $c = 0, l = -\beta^{-1}$ and that k is arbitrary. If $\lambda = 0$ then d is arbitrary and if $\lambda \neq 0$ then $d = 0$. (II) If $(a, b) = (a_i, 0)$, for $i = 1, 2$, then $c = l = 0$ and $k = (\beta a_j)^{-1}$, where $a_j \neq a_i$. If $\lambda = a_j$ then d is arbitrary and if $\lambda \neq a_j$ then $d = 0$.

Now let

$$S = \{(\theta^*, \lambda) \in \mathbb{R}^7 | T(\theta^*) = \theta^*\}.$$

We see that $S = S_G \cup S_{M,1} \cup S_{M,2} \cup S_{CF,1} \cup S_{CF,2}$, where

$$S_G = \{(\theta, \lambda) \in S | a = \beta^{-1}, b = -\beta^{-1}\delta, c = 0, l = -\beta^{-1}, \lambda = 0\},$$

$$S_{M,i} = \{(\theta, \lambda) \in S | a = a_i, b = c = 0, d = 0, k = (\beta a_j)^{-1}, l = 0, \lambda \neq a_j\}$$

$$S_{CF,i} = \{(\theta, \lambda) \in S | a = a_i, b = c = 0, k = (\beta a_j)^{-1}, l = 0, \lambda = a_j\}.$$

We have the following proposition.

Proposition 5 Let $X_t = [y_{t-1}, y_{t-2}, 1, \xi_t, v_t, v_{t-1}]'$, and assume that the roots a_1, a_2 are real.

1. If $\theta \in S_G$ then for any martingale difference sequence ε_t , $y_t = \theta' X_t$ is a general form representation.
2. If $\theta \in S_{M,i}$ or $\theta \in S_{CF,i}$ then $y_t = \theta' X_t$ is a common factor representation.
3. If y_t is an REE then
 - (a) there exists $\theta \in S_G$ and martingale difference sequence ε_t so that $y_t = \theta' X_t$.
 - (b) for $i = 1, 2$ there exists $\theta \in S_{M,i} \cup S_{CF,i}$ and martingale difference sequence ε_t so that $y_t = \theta' X_t$.

Given the results from Propositions 3 and 4, the only potential subtlety in proving this Proposition is the observation that if a common factor solution conditions on the process ξ_t then there is a solution conditioning on $d\xi_t$ for any d . To see this, notice that if ξ_t satisfies (8), then $\xi'_t = d\xi_t$ satisfies

$$\xi'_t = a_j \xi'_{t-1} - (\beta a_j)^{-1} v_t + \varepsilon'_t,$$

where $\varepsilon'_t = d\varepsilon_t - (\beta a_j)^{-1}(1+d)v_t$.

4 E-stability

Let θ^* be a fixed point of the T-map. We say θ^* (and the associated REER) is *E-stable* provided the differential equation

$$\frac{d\theta}{d\tau} = T(\theta) - \theta \tag{17}$$

is locally asymptotically stable at θ^* . The *E-stability Principle* says that if the REER is E-stable then it is learnable by a reasonable adaptive algorithm. This principle is known to be valid for least squares and closely related statistical learning rules in a wide variety of models. For a thorough discussion see (Evans and Honkapohja 2001).⁸

⁸The connection between least squares learning and E-stability is established using convergence results from the stochastic approximation literature. This technique was first applied in the macroeconomics learning literature by (Marcet and Sargent 1989).

The definition of expectational stability just given is inadequate when there is a non-trivial connected set of rest points of the differential equation (17), as is the case for our model; if the λ -section of S is locally connected then no point in S is locally asymptotically stable. In this case the notion of E-stability is extended as follows: we say that a set of fixed points, Q , is E-stable provided there is a neighborhood U of Q so that for any $\theta_0 \in U$ the trajectory of θ determined by the differential system (17) converges to a point in Q . A necessary condition for E-stability of Q is that for all points $q \in Q$ the non-zero eigenvalues of the derivative $T(\theta) - \theta$ evaluated at q have negative real part. In our case this necessary condition is also sufficient because the derivative of the T-map has only one zero eigenvalue; see “Proof of Proposition 7” in the Appendix for details. Let S_* represent any one of the five sets $S_G, S_{M,1}, S_{M,2}, S_{CF,1}, S_{CF,2}$. Abusing notation slightly, we will say that for fixed λ , the set S_* is E-stable provided it is E-stable as a subset of \mathbb{R}^6 .⁹ Finally, we shall say a representation is E-stable provided the associated set S_* is E-stable.

Notice that since a given REE can have different representations, we should not strictly speak of learnable REEs, but instead of whether an REE is learnable. For example we will find that, for some regions of the parameter space, common factor representations of an REE are stable under learning even when the general form representation is not. It will be terminologically convenient for us to describe this by saying that the common factor solution is stable while the general form solution is not. As we will see, there are also cases in which all representations of an REE are unstable.

4.1 Instability of General Form Solution

We begin with a result that the general form solutions, i.e. REE written in their general form representation, are not E-stable.

Let $\phi = [a, b]'$. Since the subsystem of the differential system (17) corresponding to ϕ is decoupled from the rest of the differential system, it follows that a necessary condition for E-stability of the general form solution is that the subsystem

$$\frac{d\phi}{d\tau} = T_\phi(\phi) - \phi \tag{18}$$

⁹Since λ is exogenous and assumed known, stability in its dimension is not relevant. If λ were not known it could be consistently estimated by a regression of ξ_t on ξ_{t-1} and E-stability conditions would be unaffected.

be locally asymptotically stable at $\phi^* = (\beta^{-1}, -\delta\beta^{-1})$, where T_ϕ is given by (11) and (12). The derivative is given by

$$D(T_\phi) - I_2 = \begin{bmatrix} 2\beta a - 1 & \beta \\ \beta b & \beta a - 1 \end{bmatrix},$$

where I_2 is the 2×2 identity matrix. Recall that a fixed point is locally asymptotically stable provided the eigenvalues of the derivative have negative real part, or, equivalently, the derivative has negative trace and positive determinant. Evaluated at the fixed point ϕ^* we have $D(T_\phi) - I_2 = \begin{bmatrix} 1 & \beta \\ -\delta & 0 \end{bmatrix}$, which has positive trace thus yielding the following result:

Proposition 6 *The general form solution is not E-stable.*

4.2 Stability of Common Factor Solutions

We now consider E-stability of CF solutions, i.e. REE written in their Common Factor representation and given by $S_{M,1} \cup S_{M,2} \cup S_{CF,1} \cup S_{CF,2}$. We again start by considering the subsystem (18). Evaluated at $\phi \equiv \bar{\phi}_i = (a_i, 0)$ we obtain $D(T_\phi) - I_2 = \begin{bmatrix} 2\beta a_i - 1 & \beta \\ 0 & \beta a_i - 1 \end{bmatrix}$, which yields the stability condition $2\beta a_i < 1$. From the expressions for a_1 and a_2 , it is immediate that this stability condition is satisfied for a_1 and fails for a_2 . Thus any REER in $S_{M,2} \cup S_{CF,2}$ is not E-stable.

For the MSV solution $S_{M,1}$ and the CF stationary sunspot equilibria $S_{CF,1}$ the remaining E-stability conditions are obtained in the Appendix. We have the following result:

Proposition 7 *The Common Factor solutions $S_{M,2} \cup S_{CF,2}$ are not E-stable. The conditions for E-stability of the Common Factor Solutions $S_{M,1}$ and $S_{CF,1}$ are as follows:*

1. *The MSV solution $S_{M,1}$ is E-stable when (i) $\beta < \frac{1}{2}$ or $\beta + \delta < 1$ and (ii) $\beta(a_1 + \lambda) < 1$,*
2. *The CF solution set $S_{CF,1}$ is E-stable when $\beta < \frac{1}{2}$ or $\beta + \delta < 1$.*

Figure 1 illustrates the region in (β, δ) space in which the Common Factor solution SSEs are E-stable. The region is obtained by simultaneously imposing the conditions for indeterminacy, that the roots be real, and that conditions for E-stability of the CF solutions be satisfied. The resulting region is specified by $\beta < -\frac{1}{2}$, $\beta + \delta < -1$ and $4\beta\delta < 1$.

FIGURE 1 ABOUT HERE

We also make several remarks concerning the E-stability condition (ii) for the MSV solution $S_{M,1}$. Recall that the CF solution set $S_{CF,1}$ requires $\lambda = a_2$. Since $\beta(a_1 + a_2) = 1$, condition (ii) is not (quite) satisfied when the CF solution set is E-stable. However, note that the CF solution set $S_{CF,1}$ includes a continuum of values for d and that the CF solution for $d = 0$ describes an REE that is identical to the corresponding MSV solution.

The other point to note in connection with condition (ii) for the MSV solution is that it would not be required if the PLM (9) excluded the variable ξ_t . In that case E-stability of $S_{M,1}$ is governed by (i) and it can be seen that this condition is always satisfied throughout the regular region $|\beta + \delta| < 1$, as well as in parts of the irregular and explosive regions. Condition (ii) provides an additional requirement that guarantees stability of this MSV solution even if agents condition also on ξ_t . Further discussion is given in the Appendix.

4.3 Discussion

Our results show that stationary sunspot equilibria are E-stable in a subset of the indeterminacy (or “irregular model”) region. This is true even though the set of general form representations (6), which take an ARMA(2,1) form, always fail to be E-stable in this model.¹⁰ The key to obtaining E-stability in this subregion is that agents use a perceived law of motion in which they condition forecasts, not only on the predetermined variable, but also on an exogenous AR(1) process ξ_t with autoregressive parameter $\lambda = a_2$. This exogenous process can be thought of as having a “resonant frequency,” i.e. having time-series properties that are exactly right for exciting a stationary sunspot equilibrium. When the conditions $\beta < -\frac{1}{2}$, $\beta + \delta < -1$ and $4\beta\delta < 1$ are satisfied, conditioning learning on ξ_t leads to an E-stable set of SSEs.

¹⁰In some linear models the general form solutions can be E-stable. This was found in (Evans and Honkapohja 1992), (Duffy 1994) and (Evans and Honkapohja 1994a) for models that rely on different information sets and include forecasts over multiple horizons.

These SSEs have the same time-series properties as the ARMA(2,1) solutions, because they are an alternative representation of these solutions.

The term “resonant frequency” was used in (Evans and Honkapohja 2002b) to describe the (well-known) condition on transition probabilities required for existence of finite state Markov SSEs in forward-looking models. Here we see that the phenomenon is in fact much more general and applies also to models with predetermined variables. An exogenous AR(1) process with an autocorrelation parameter equal to one of the two roots can form the basis of a sunspot equilibrium for a PLM that conditions on this exogenous variable as well as its own lag. These are the CF representations of SSEs. Furthermore, in the subset of the irregular region shown in Figure 1, conditioning on the resonant frequency sunspot variable $\xi_t = a_2\xi_{t-1} + \varepsilon_t$ leads to SSEs that are E-stable and are therefore (as we verify below) stable under least-squares learning.

5 Simulations of Real-Time Learning

Our conjecture is that for values of β and δ in the indicated region of Figure 1, the corresponding CF solutions are stable under least squares learning. We have shown that in this area the conditions for E-stability are satisfied for the common factor representation of SSEs. The E-stability principle says that if an REER is E-stable then it is learnable by least squares and closely related learning algorithms. However, the E-stability principle is only known to formally apply to REER which are isolated rest points of the differential equation (17). To support our focus on E-stability we now simulate least squares learning of the REER.

Before presenting the numerical results we briefly outline the framework under which the simulations are generated. Under least squares learning agents are assumed to have a PLM in which y_t depends linearly on predetermined and exogenous variables X_t with coefficients θ . The values of θ are unknown to the agents, who therefore use an estimate of its value. More specifically, we assume that agents make forecasts based on the PLM (9). At the beginning of period t agents have available observations of $X_t = [y_{t-1}, y_{t-2}, 1, \xi_t, v_t, v_{t-1}]'$ and use the estimate θ_{t-1} of $\theta = [a, b, c, d, k, l]'$ based on data through period $t - 1$. Their forecasts are thus given by (10) with θ replaced by θ_{t-1} . The value of y_t is then determined according to (1) by these expectations and by y_{t-1} and v_t . The outcome can be conveniently

expressed as $y_t = T(\theta_{t-1})'X_t$ where $T(\theta)$ is specified by (11)-(16).

We then move to the next period $t + 1$, at the beginning of which agents update their least squares estimate of θ using the time t data point. The recursive least squares (RLS) learning algorithm¹¹ for the updated coefficients of the least squares regression of y_t on X_t is given by

$$\theta_t = \theta_{t-1} + \frac{1}{t}R_t^{-1}X_t(y_t - \theta_{t-1}'X_t), \quad (19)$$

$$R_t = R_{t-1} + \frac{1}{t}(X_tX_t' - R_{t-1}). \quad (20)$$

Note that $y_t - \theta_{t-1}'X_t$ is the period t residual for the estimated relationship using the previous period's estimate θ_{t-1} . Using $y_t = T(\theta_{t-1})'X_t$ the θ_t recursion can be written as

$$\theta_t = \theta_{t-1} + \frac{1}{t}R_t^{-1}X_t((T(\theta_{t-1}) - \theta_{t-1})'X_t). \quad (21)$$

The sequence of events is then again repeated: exogenous variables v_{t+1} and ξ_{t+1} are then determined, agents form expectations and y_{t+1} is determined. The question of interest is whether $\theta_t \rightarrow \theta \in S$ as $t \rightarrow \infty$. In particular if $\theta_t \rightarrow \theta \in S_G \cup S_{CF,1} \cup S_{CF,2}$ then the process y_t converges asymptotically to an SSE.

Note that we have chosen the functional form of the PLM (9) so that the associated T-map has as fixed points all REER of interest. The generality of this form implies that for a given representation the PLM may be over-parameterized so that the associated fixed point has zeros in some entries. This over-parameterization poses no problem for E-stability; however, there may be important consequences for the corresponding real time learning algorithm. Specifically, if the over-parametrization leads to severe multicollinearity, the statistical learning algorithm may settle on a vector of parameters that is not itself a fixed point of the T-map, but which provides an equivalent, overparameterized representation of a fixed point. When this occurs, the associated sequence of endogenous variables appears to converge to the corresponding REE (at least up to the point at which severe multicollinearity leads to numerical breakdown of the algorithm). We begin by analyzing this phenomenon in more detail, and show that it can be sidestepped in a natural way by choosing regressors that do not yield multicollinearity problems.

¹¹See, for example pp. 32-3 of (Evans and Honkapohja 2001).

We then turn to the main focus of this Section: numerical verification that E-stability governs convergence of least squares learning algorithms, including instability of general form solutions and possible convergence to common factor sunspot solutions.

5.1 Over-Parameterization and Multicollinearity

The overparameterization issue is most easily examined by considering representations without sunspots. Specifically, we assume the following PLM:

$$y_t = ay_{t-1} + by_{t-2} + kv_t + lv_{t-1}. \quad (22)$$

The associated T-map is given by (11), (12), (15), and (16). A fixed point of this T-map is given by $a = a_1$, $b = 0$, $k = (1 - \beta a_1)^{-1}$, and $l = 0$, yielding a representation of the form

$$y_t = a_1 y_{t-1} + (1 - \beta a_1)^{-1} v_t. \quad (23)$$

Arguments already given show that for appropriate parameters, this representation is E-stable; however, numerically, we find that real time learning algorithms can settle on parameter values other than those yielding this representation. For example, setting the model's parameter values to $\beta = -3$ and $\delta = 1$, and using the above PLM, we obtained a simulation of least squares learning yielding

$$(a_t, b_t, k_t, l_t) \longrightarrow (.0445, .1693, .4309, .1694),$$

which contrasts with the stable MSV values of $(.4343, 0, .4343, 0)$. Other simulations converge to different parameter values, or exhibit explosive behavior resulting from the instability of the general form representation: see section 5.2 for details.¹²

To understand these results, first notice that if the process y_t solves (23) and $\eta \in \mathbb{R}$ then y_t also solves

$$(1 - \eta L)(1 - a_1 L)y_t = (1 - \beta a_1)^{-1} (1 - \eta L)v_t. \quad (24)$$

This ARMA(2,1) process is consistent with the agents' PLM. If y_t solves (23) and agents believe in the ARMA(2,1) process then their regression residuals

¹²The simulations were conducted using an RLS algorithm with a random initial condition. The values reported are those obtained after 10,000 periods, though the estimates were very close to these values after approximately 500 periods.

are zero. (See Lemma 8 below.) Since RLS learning relies on these residuals to update the estimator, it follows that in this case the estimator will be stuck at a point which does not correspond to a REER. Note, however, that the process y_t is an REE. For the simulation mentioned above, $\eta = -.3901$ yields the ARMA(2,1) representation of the MSV solution.

Lemma 8 *Let $\eta \in \mathbb{R}$, T be the T -map associated with the PLM (22), and let*

$$\begin{aligned}\theta &= [a_1 + \eta, -a_1\eta, (1 - \beta a_1)^{-1}, -\eta(1 - \beta a_1)^{-1}]', \\ X_t &= [y_{t-1}, y_{t-2}, v_t, v_{t-1}]'. \end{aligned}$$

If y_t satisfies (23) then $(T(\theta) - \theta)' X_t = 0$.

Intuitively this Lemma must hold: if y_t satisfies (23) then y_t satisfies (24); if agents know y_t satisfies (24) then agents forecast y_t correctly; if agents forecast y_t correctly then $T(\theta)' X_t = \theta' X_t$. The formal proof is straightforward but tedious algebra and is omitted.

The central problem with the over-parameterization is that it leads to severe multicollinearity. Notice that if y_t solves (23) then the regressors in the ARMA(2,1) representation display perfect linear correlation asymptotically; y_{t-1} is a linear combination of y_{t-2} and v_{t-1} .¹³ To avoid this problem, it is natural to follow standard econometric practice and drop one or more explanatory variables. For example, if the regressors are taken to be y_{t-1} , y_{t-2} , and v_t , then convergence of the RLS estimator to the MSV solution (23) does obtain.

5.2 Instability of General Form Solutions

Recall that general form solutions are given by

$$y_t = \beta^{-1} y_{t-1} - \delta \beta^{-1} y_{t-2} - \beta^{-1} v_{t-1} + \varepsilon_t,$$

where ε_t is any martingale difference sequence and is taken to be observed. The form of the model requires v_t to be a regressor in the agent's PLM, even though it is not present in the general form solution. (We are thus forced to

¹³Indeed if one extends the simulation shown in Figure 2 for a sufficiently long period, the matrix R_t eventually becomes close enough to singularity to result in numerical breakdown of the algorithm.

specify a PLM that can sometimes lead to severe multicollinearity problems.) Specifically, we assume a PLM of the form (9) with $\lambda = 0$ so that $\xi_t = \varepsilon_t$:

$$y_t = ay_{t-1} + by_{t-2} + c + d\varepsilon_t + kv_t + lv_{t-1}. \quad (25)$$

Set $X_t = [y_{t-1}, y_{t-2}, 1, \varepsilon_t, v_t, v_{t-1}]'$, and $\theta = [a, b, c, d, k, l]'$. The behavior of the algorithm (21) and (20) was analyzed via simulations. The algorithm was initialized by choosing point at random within a given neighborhood of the set S_G .

As predicted by the E-stability principle, convergence of the real time estimators to the parameters associated with the general form solution does not appear to obtain; all simulations were consistent with this REER being unstable under least squares learning. Interestingly, in cases when the norms of the estimates did not diverge to infinity, the estimators appeared to settle on seemingly arbitrary values, with the exception of the estimator for k , which appears to converge to $(1 - \beta a_1)^{-1}$, and c and d , which appear to converge to zero. To explain this behavior, we must turn back to the problem of over-parameterization. As mentioned above, to nest the various representations we must over-parameterize the PLM. If convergence to the general form solution obtains then the problem of severe multicollinearity will not arise; however, the general form solution is not stable and the algorithm quickly moves the estimates away from these values. As the estimates θ_t adjust, so too does the endogenous variable y_t ; if y_t adjusts so that it follows the common factor representation

$$y_t = a_1 y_{t-1} + (1 - \beta a_1)^{-1} v_t,$$

then, as was shown in the previous subsection, the estimators will settle on an ARMA(2,1) process which is consistent with this AR(1) representation, that is, the agents will learn an over-specified model. In each case in which a simulation resulted in convergence to some fixed values for the estimates, the associated ARMA(2,1) process was an over-parameterization of the AR(1) common factor solution produced above. Using the PLM (25), and with $\beta = -3$, and $\delta = 1$ an example simulation of least squares learning results in

$$(a_t, b_t, c_t, d_t, k_t, l_t) \longrightarrow (-.1490, .2559, 0, 0, .4202, .2552);$$

a common factor value of $\eta = -.5877$ then yields the MSV solution. We conclude that although the general form REER is not stable under RLS learning, the agents may still learn an REE. Also, the learned REE will not depend on the sunspot ε_t .

5.3 Stability of CF Solutions

Here we analyze numerically the real-time learning implications of Proposition 7. Recall that E-stable common factor representations have the form

$$\begin{aligned} y_t &= a_1 y_{t-1} + d\xi_t + (1 - \beta a_1)^{-1} v_t, \\ \xi_t &= \lambda \xi_{t-1} + \varepsilon_t, \end{aligned}$$

where λ is taken as known.¹⁴ We implement real-time learning by assuming a PLM of the form

$$y_t = a y_{t-1} + c + d\xi_t + k v_t + l v_{t-1}.$$

We have dropped the regressor y_{t-2} to avoid problems with multicollinearity. Setting $\theta = [a, c, d, k, l]'$ and $X_t = [y_{t-1}, 1, \xi_t, v_t, v_{t-1}]'$ results in an updating algorithm of the form (21) and (20). We consider the sets $S_{M,1}$ and $S_{CF,1}$ separately.

5.3.1 MSV-Solutions

The E-stability Principle, together with Proposition 7, predicts that if $\beta < 1/2$, or $\beta + \delta < 1$, and if $\beta(a_1 + \lambda) < 1$, then the RLS learning algorithm described above should converge locally to the MSV solution. To test this principle, we chose different parameter values satisfying these conditions and simulated the learning model using standard normal white noise for the martingale difference sequence ε_t . Initial conditions were chosen randomly within a small neighborhood of the MSV parameter values.

Analytic results implying convergence with probability one typically require amending the algorithm with a projection facility. Alternatively, one can adjust the gain of the algorithm to obtain convergence with probability approaching one. For the simulations produced here we scale the gain of the RLS algorithm by 1/25, thus increasing the probability of convergence.

We found that with positive probability (that is, for a positive proportion of our simulations), convergence to the MSV solution appeared to obtain. An example simulation corresponding to the parameter values $\beta = -3$, $\lambda = .5$, and $\delta = 1$ (so that $\beta(a_1 + \lambda) = -2.8$ approximately) showed convergence to the MSV solution, with b_t and d_t going rapidly to zero. Convergence of a_t ,

¹⁴The assumption that λ is known is made for simplicity. If λ is unknown it can be estimated by agents and the stability conditions are unchanged.

k_t , and l_t to .4343, .4343, and 0, respectively, took somewhat longer, with small movements still occurring after 7000 periods. This behavior is due in part to the scaled gain used to promote convergence.

5.3.2 Common Factor Solutions

We used simulations analogous to those mentioned above to analyze the stability of common factor sunspots for the case $\lambda = a_2$. Initial conditions were chosen at random near the set $S_{CF,1}$; the initial value of θ_3 was chosen at random near one. Proposition 7 predicts that if $\beta < 1/2$ or $\beta + \delta < 1$ then convergence to some point in $S_{CF,1}$ will obtain. Our simulations suggest that with positive probability, this is indeed the case.

FIGURE 2 ABOUT HERE

Figure 6 shows the results of a simulation corresponding to the parameter values $\beta = -3$, and $\delta = 1$. Notice, in particular, that $\theta_3 = d$ appears to converge to some number not equal to zero, suggesting that agents in this economy will indeed learn that there is a dependence on a sunspot variable.

6 Further Discussion and Extensions

We have considered a simple reduced form model with one lead, one lag and white noise shocks, but it is one that arises from some standard structural models. For example, the investment model of (Lucas and Prescott 1971) as extended in (Sargent 1987), Ch. XIV, to allow for market distortions due to taxes and externalities, yields the reduced form we considered. This model examines the behavior of a large number of identical firms in a competitive industry. Firms face a stochastic demand and quadratic capital adjustment costs, thus giving the model its forward looking component. With no externalities or taxes there is a unique stationary REE. In general, however, the model's reduced form parameters are unrestricted. For some specifications, the model supports stable common factor sunspots. The possibility that SSEs in this model can be stable under learning has not previously been noticed. Our results suggest that research into the empirical plausibility of

these cases would be of considerable interest.¹⁵

In our exploration of stability conditions, we have assumed that, when forming expectations $E_t^*y_{t+1}$ using the PLM (9), agents have available observations of current exogenous variables v_t and ξ_t and the lagged endogenous variable y_{t-1} . An alternative assumption is that expectations $E_t^*y_{t+1}$ can condition also on current y_t . In this case y_t and $E_t^*y_{t+1}$ are simultaneously determined. Although the assumptions are equivalent under rational expectations they lead to different out of equilibrium behavior and therefore can lead to different stability conditions under learning. We analyzed the stability of CF solutions in the indeterminate case under the assumption that agents condition on current information. For brevity, we omit the details of the analysis. Up to minor modification, the techniques that applied to the previous case also apply here.

We find that the E-stability conditions for the set $S_{CF,1}$ are unchanged: this solution set is E-stable if and only if $\beta < \frac{1}{2}$ or $\beta + \delta < 1$ and thus is E-stable as before in the shaded region marked in Figure 1. However, a new case arises with the set $S_{CF,2}$. Under the earlier information assumptions this set was never E-stable, but under the current assumptions it is E-stable in the region bounded by $\delta < 0$ and $\beta + \delta > 1$. We continue to find that neither CF solution set is E-stable for the part of the irregular region in which $\beta, \delta > 0$.

These results confirm (and somewhat strengthen) our earlier results: for a strict subset of the parameter space there exist SSEs that are locally stable under learning provided agents use a PLM that nests the common factor representation of these SSEs.

7 Conclusion

We have studied in detail the stability under learning of stationary sunspot equilibria (SSEs) in a linear stochastic univariate expectations model with a predetermined variable. In the irregular case in which SSEs exist there are alternative representations of the solutions, and we have shown that the form of the representation is crucial for the stability of SSEs under learning.

¹⁵For this model the stability under learning of the Minimal State Variable solutions was examined in (Evans and Honkapohja 2001), Section 8.6.2. Also, a more detailed discussion of this model is contained in (Evans and McGough 2002), including analysis of stable k -state sunspots.

An SSE can be represented (in “general form”) as an ARMA(2,1) process depending on an exogenous martingale difference sequence. When agents use such representations to estimate the law of motion to generate their forecasts, none of the SSEs are stable under learning. However, SSEs have an alternative “common factor” representation, and when agents use this perceived law of motion, the SSEs are locally stable under learning in a strict subset of the parameter space. We have illustrated and extended our theoretical results using numerical simulations. In the Introduction we emphasized the importance in applied macroeconomics of multivariate expectations models with predetermined variables. Given their potential importance to business cycle theory, the extension of the findings of this paper to multivariate models is a high priority for our current research.

8 Appendix

8.1 Stationarity and Boundedness in Mean

We here carefully state and prove the results described in Section 2. We begin with notation and definitions.

A stochastic process is a countable collection of random variables y_t defined on a given probability space $(\Omega, \mathcal{F}, \mu)$. If the process is defined for $t \geq t_0$ we say that the process is initialized, otherwise we say the process is doubly infinite. Given the process y_t , define $\mathcal{F}_t \subset \mathcal{F}$ to be the smallest σ -field so that y_s is measurable for all $s \leq t$. Note that $\mathcal{F}_t \subset \mathcal{F}_{t+1}$. We now recall the definition of conditional expectation: $E_t y_{t+s} : \Omega \rightarrow \mathbb{R}$ is the almost everywhere unique \mathcal{F}_t -measurable function so that for all $\Lambda \in \mathcal{F}_t$ we have

$$\int_{\Lambda} y_{t+s} d\mu = \int_{\Lambda} E_t y_{t+s} d\mu.$$

We now make the following definitions: We say that the process y_t is *Uniformly Bounded in Expectation* (UB) provided

$$\sup_t E|y_t| < \infty.$$

We say the process is *Conditionally Uniformly Bounded in Expectation* (CUB) provided that for all t

$$\sup_s E_t |y_{t+s}| < \infty \quad a.e.$$

We have the following lemma.

Lemma 9 *If the process is uniformly bounded in expectation then it is conditionally uniformly bounded in expectation.*

Proof. We prove the contra-positive. Assume the process is not CUB. WLOG, $y_n \geq 0$. Then for some n we have that at least one of the following two statements must hold: $\exists B \in \mathcal{F}_n$ of positive measure such that

1. $\lim_{s \rightarrow \infty} E_n y_{n+\tau(s)} = \infty$
2. $\lim_{s \rightarrow \infty} E_n y_{n-\tau(s)} = \infty$

on B for some subsequence as defined by the monotonic increasing function $\tau : \mathbb{N} \rightarrow \mathbb{N}$. In the first case, we proceed as follows: that (1) holds implies that for any positive real numbers M and δ , there exists \mathcal{F}_n -measurable $\hat{B} \subset B$ and positive integer N so that $\mu(\hat{B}) > \mu(B) - \delta$ and $E_n y_{n+\tau(s)} \geq M$ on \hat{B} for all $s \geq N$. Now notice $s \geq N$ implies

$$E(y_{n+\tau(s)}) \geq \int_{\hat{B}} y_{n+\tau(s)} d\mu = \int_{\hat{B}} E_n y_{n+\tau(s)} d\mu \geq M(\mu(B) - \delta)$$

where the middle equality follows from the definition of the conditional expectations operator. Since δ was arbitrarily small and M was arbitrarily large, we have that the unconditional expectation is unbounded. To address the second case simply notice the same argument holds. Of course, in this case, for $s \leq 0$, $E_n y_{n+s} = y_{n+s}$. ■

The converse to the above result is false. As a simple example, let x be any positive random variable with infinite expectation: for example, let z be uniformly distributed on $(0, 1)$ and let $x = 1/(1 - z)$. Now let $y_t = x$. Then $E_t y_{t+s} = x$ so that the process is CUB, but $E y_t = \infty$ so that the process is not UB. Notice this argument applies to both initialized and doubly infinite processes.

We may now extend the comparison to stationary and asymptotically stationary processes. Let $\{y_t\}$ be a stochastic process and γ_{jt} the j^{th} -autocovariance of y_t . If y_t is doubly infinite then it is covariance stationary (CS) provided $E y_t$ and γ_{jt} exist and are independent of time for all t and j . If y_t is an initialized process then it is asymptotically covariance stationary (ACS) provided $E y_t$ and γ_{jt} exist and the following hold:

1. $\lim_{s \rightarrow \infty} E y_s < \infty$
2. $\lim_{s \rightarrow \infty} \gamma_{js} < \infty \forall j \in \mathbb{N}$.

The second condition in particular requires that $\lim_{s \rightarrow \infty} \gamma_{0s} < \infty$. It is well known that $E |x|^p < \infty$ implies $E |x|^{p-1} < \infty$ for any random variable x and for all $p > 1$. Hence $\lim_{s \rightarrow \infty} \gamma_{0s} < \infty$ implies $\lim_{s \rightarrow \infty} E |y_s| < \infty$. Thus CS or ACS implies UB. To see that UB does not imply CS or ACS, consider the following deterministic process

$$y_t = \begin{cases} 1 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}$$

This process is clearly UB and neither stationary nor asymptotically stationary. We summarize the above results in the following proposition.

Proposition 10 1. If y_t is a doubly infinite process then

- (a) CS implies UB implies CUB.
- (b) CUB does not imply UB does not imply CS.

2. If y_t is an initialized process then

- (a) ACS implies UB implies CUB.
- (b) CUB does not imply UB does not imply ACS.

8.2 Proof of Proposition 1

Any solution of (1) can be written in the form (5). Considering first the regular case, we can factor A as $A = Q \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} Q^{-1}$, where the columns of Q are the eigenvectors of a_1 and a_2 respectively. It can be verified that $Q = \begin{pmatrix} a_1 & a_2 \\ 1 & 1 \end{pmatrix}$ and $Q^{-1} = (a_1 - a_2)^{-1} \begin{pmatrix} 1 & -a_2 \\ -1 & a_1 \end{pmatrix}$. Letting $Q^{-1} = \begin{pmatrix} q^{11} & q^{12} \\ q^{21} & q^{22} \end{pmatrix}$ and defining $\begin{pmatrix} p_{t+1} \\ r_{t+1} \end{pmatrix} = Q^{-1} \begin{pmatrix} y_{t+1} \\ y_t \end{pmatrix}$, (5) can be rewritten as

$$\begin{pmatrix} p_{t+1} \\ r_{t+1} \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} p_t \\ r_t \end{pmatrix} + \begin{pmatrix} -q^{11}\beta^{-1} \\ -q^{21}\beta^{-1} \end{pmatrix} v_t + \begin{pmatrix} q^{11} \\ q^{21} \end{pmatrix} \varepsilon_{t+1}.$$

It follows that

$$\begin{aligned} E_t p_{t+1} &= a_1 p_t - q^{11} \beta^{-1} v_t \text{ and } E_t p_{t+s+1} = a_1 E_t p_{t+s} \text{ for } s = 1, 2, 3, \dots \\ E_t r_{t+1} &= a_2 r_t - q^{21} \beta^{-1} v_t \text{ and } E_t r_{t+s+1} = a_2 E_t r_{t+s} \text{ for } s = 1, 2, 3, \dots \end{aligned}$$

Since $|a_2| > 1$ and $E_t r_{t+s} = a_2^{s-1} E_t r_{t+1}$ we have $\lim_{s \rightarrow \infty} |E_t r_{t+s}| = \infty$ unless $r_t = a_2^{-1} q^{21} \beta^{-1} v_t$. It follows that a nonexplosive solution must satisfy $y_t = -(q^{22}/q^{21})y_{t-1} + a_2^{-1} \beta^{-1} v_t$. Since $-(q^{22}/q^{21}) = a_1$ we have $y_t = a_1 y_{t-1} + (\beta a_2)^{-1} v_t$. There is a unique such solution that meets the initial condition and a unique such stationary solution. Therefore (ii) follows.

In the case $|a_1|, |a_2| > 1$, suppose that the roots are real. Then we still have $|a_2| > 1$ so that imposing the associated nonexplosiveness condition implies $y_t = a_1 y_{t-1} + (1 - \beta a_1)^{-1} v_t$. However $|a_1| > 1$ implies that this solution is also explosive, and hence there are no nonexplosive solutions,

almost surely. The complex case can be handled by using a decomposition of A as $WrR_\omega W^{-1}$ with the roots of A being represented by $r(\cos \omega \pm i \sin \omega)$, associated eigenvectors $m \pm in$, $R_\omega = \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix}$ and $W = \begin{pmatrix} n & m \end{pmatrix}$.

In the irregular case $|a_1|, |a_2| < 1$, all solutions of the form (4) are non-explosive. For ε_{t+1} a stationary stochastic process the solution

$$y_{t+1} = (1 - \beta^{-1}L - \beta^{-1}\delta L^2)^{-1}(-\beta^{-1}v_t + \varepsilon_{t+1})$$

is stationary. If an initial condition $y_0 = \check{y}_0$ is given then all solutions of the form (4) for $t = 1, 2, 3, \dots$, are nonexplosive, where y_1 is arbitrary. Letting $x_{t+1} = (y_{t+1}, y_t)'$ we can rewrite (4) as $x_{t+1} = Ax_t - \beta^{-1}v_t + \varepsilon_{t+1}$ and hence $x_{t+1} = A^t x_1 + \sum_{j=0}^{t-1} A^j (-\beta^{-1}v_{t-j} + \varepsilon_{t+1-j})$, where $x_1 = (y_1, \check{y}_0)'$. Consider another solution generated by the same martingale difference sequence ε_{t+1} but with initial condition $x'_1 = (y'_1, \check{y}_0)'$ with $y'_1 \neq y_1$. Then $|x'_{t+1} - x_{t+1}| = |A^t(x'_1 - x_1)| \rightarrow 0$. Hence every solution is indeterminate and (iii) is established.

8.3 E-stability Conditions for CF Solutions

8.3.1 Proof of Proposition 7

The main text shows that in the differential equation for (a, b) the fixed point $(a_2, 0)$ is locally unstable and hence that $S_{M,2}$ and $S_{CF,2}$ are E-unstable. The main text also shows that $(a_1, 0)$ is locally stable. We now obtain the additional E-stability conditions for $S_{M,1}$ and $S_{CF,1}$. The differential equations in (a, b, c) have a recursive structure since a and b are independent of c . The equation in c thus gives the additional stability condition $\beta(1 + a_1) < 1$, which is equivalent to $2\beta - 1 < \sqrt{1 - 4\beta\delta}$. Clearly this is satisfied for $\beta < \frac{1}{2}$ and for $\beta \geq \frac{1}{2}$ it is satisfied if $(2\beta - 1)^2 < 1 - 4\beta\delta$, which is equivalent to $\beta + \delta < 1$. The differential equations in (a, b, k, l) also have a recursive structure and since $\beta a_1 < 1$ it is easily verified that $(k, l) \rightarrow ((1 - \beta a_1)^{-1}, 0)$. This leaves the additional conditions required for convergence of the component d . Again the system (a, b, d) is recursive with a, b independent of d . For the MSV solution $S_{M,1}$, because $a \rightarrow a_1$, the additional E-stability condition is given by $\beta(a_1 + \lambda) < 1$. For the CF solution $S_{CF,1}$ we have $\lambda = a_2$ and $\beta(a_1 + \lambda) = 1$. For this case the differential equation for d is given by $d/d\tau(d(\tau)) = (\beta(a + \lambda) - 1)d(\tau)$. Using an argument analogous to (Evans and Honkapohja 1992), p. 6, it can be shown that $a(\tau) \rightarrow a_1$ implies that

$d(\tau)$ converges to a finite value, and no additional stability conditions are required.

8.3.2 Discussion of E-stability conditions for $S_{M,1}$

We discuss briefly the condition $\beta(a_1 + \lambda) < 1$ required for E-stability of the MSV solution $S_{M,1}$. Because $\beta(1 + a_1) < 1$ is also required for E-stability, and because $\lambda < 1$, the condition $\beta(a_1 + \lambda) < 1$ is satisfied whenever $\beta > 0$. In the case $\beta < 0$ we need $2\beta\lambda - 1 < \sqrt{1 - 4\beta\delta}$. It can be shown that this always holds if $\beta > -\frac{1}{2}$ or if $\beta + \delta > -1$. In the remaining region, namely $\beta < -\frac{1}{2}$ and $\beta + \delta < -1$, E-stability requires that $\lambda > a_2$.

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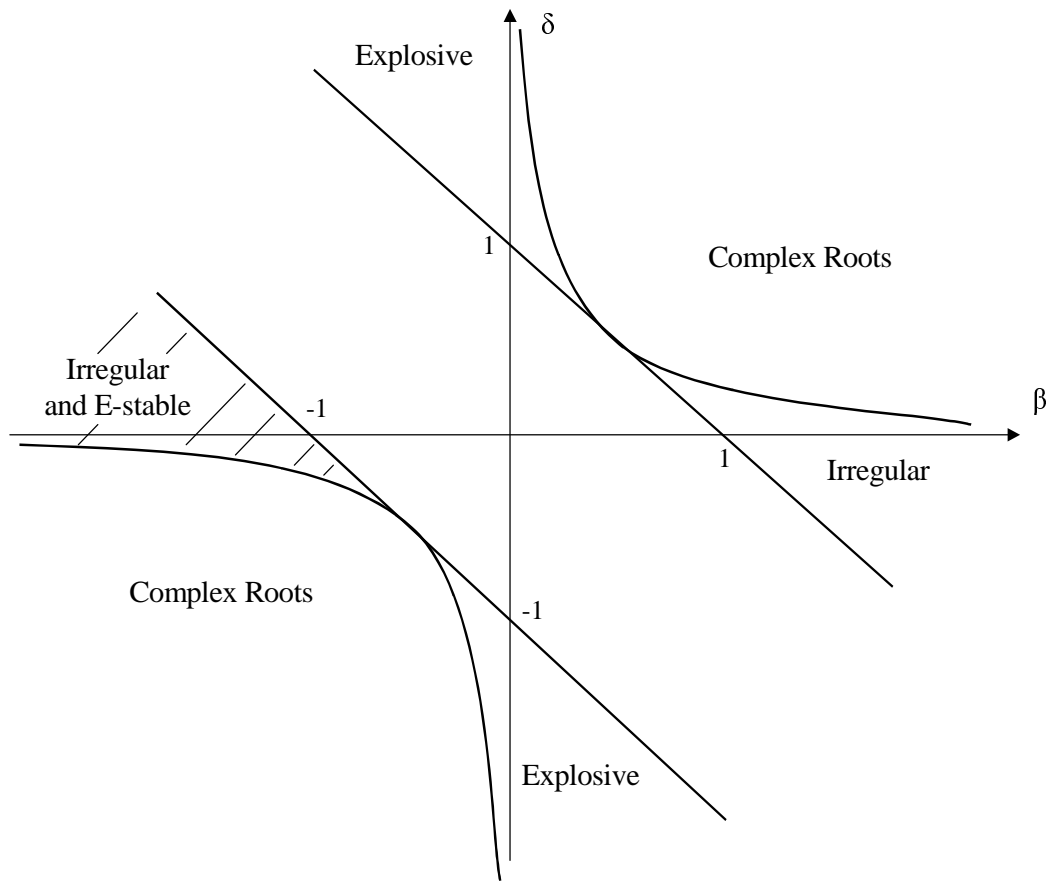


Figure 1:

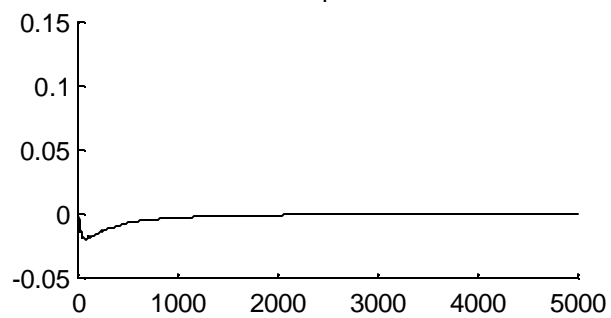
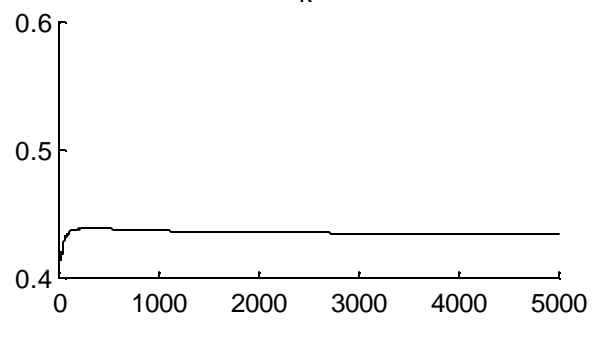
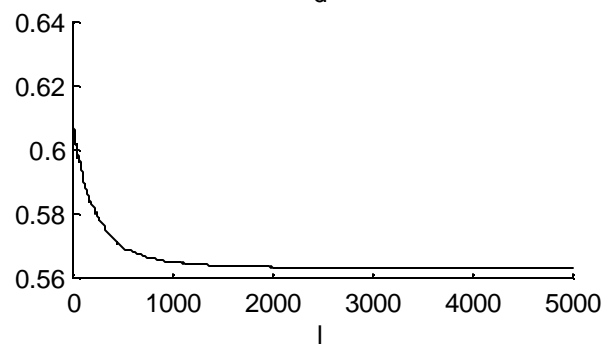
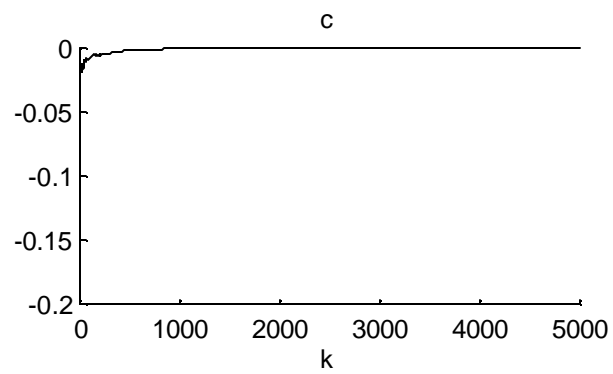
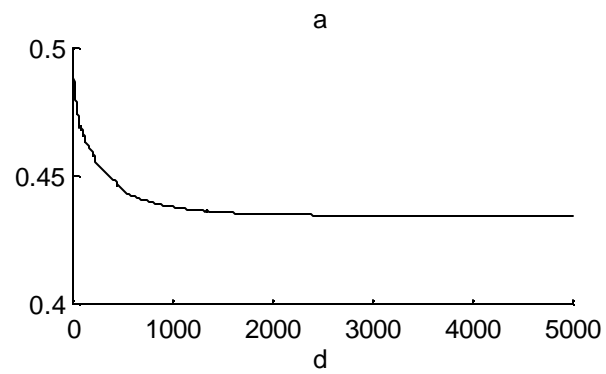


Figure 2: $\beta=-3, \delta=1, \lambda=a_2$