On Functional Form Representation of Multi-Output Production Technologies

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Abstract: The introduction of directional distance functions has given researchers an alternative to Shephard distance functions. In this paper we conduct a Monte Carlo study to determine which distance function better approximates models of technology. We conclude that quadratic representations of technology have better economic approximation properties than translog parameterizations.

Keywords and Phrases: distance functions, parameterization.

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1. Introduction

Distance functions play a central role in modeling production technologies, creating indexes and productivity measures, as well as in duality theory. With the introduction of shortage functions [Luenberger (1992)] or, as they have also been called, directional distance functions [Chambers *et al.* (1998)], researchers have been given alternatives to the Shephard input and output distance functions [Shephard (1953, 1970)]. Both sets of functions fully represent a technology but they differ in that directional distance functions meet the translation property, whereas the Shephard distance functions are homogeneous of degree one. This difference has a profound impact on what parameterization may be chosen.

In this paper we discuss various parametric forms of the two sets of distance functions. We also conduct a Monte Carlo study as a means to suggest the 'best' distance function to approximate a production technology. We find that the output set is better parameterized *via* a quadratic directional distance function than with a translog Shephard distance function.

Other papers that investigate the properties of various parametric functional forms include Gulkey *et al.* (1983), Perroni and Rutherford (1998), and Vardanyan and Noh (in press). Gulkey *et al.* examine the behavior of three functions that are used to model cost functions. They set up a Monte Carlo experiment and estimate cost functions to show that the translog functional form, although not perfect, is acceptable, as it outperforms other parameterizations, such as the generalized Leontief functional form [Diewert (1971)]. Perroni and Rutherford study the global properties of the cost function by calibrating its parameters to satisfy a particular structure of the production technology. Their results indicate that the translog functional form often violates regularity conditions. They note that better behaved parametric specifications, such as the nonseparable nested constant elasticity of substitution function [Perroni and Rutherford (1995)], are available. Finally, Vardanyan and Noh (in press) use data from the U.S. electric utility industry to show that the translog function has poor global approximation properties relative to the quadratic function.

2. Functional Forms

Let $x \in \Re_+^N$ represent inputs used to produce outputs $y \in \Re_+^M$ and T = $\{(x, y) \in \Re_+^{N+M} : x$ can produce $y\}$ denote the production set. For all $(x, y) \in T$ the technology is represented by the output sets

$$P(x) = \{y : (x, y) \in T\}. \tag{1}$$

We assume that $P(x)$ satisfies standard axioms [Färe and Primont (1995)], which include compactness and free disposability of inputs and outputs. The Shephard output distance function is defined in terms of the output sets $P(x)$ as

$$D(x, y) = \inf\{\phi > 0 : y/\phi \in P(x)\}. \tag{2}$$

The two properties of this function that are important for this paper are:

i) Homogeneity: $D(x, \mu y) = \mu D(x, y), \quad \mu > 0$, *ii) Representation:* $D(x,y) \le 1$ *if and only if* $y \in P(x)$.

The homogeneity property follows from the definition of the distance function, and the representation is due to the assumption of free disposability of outputs. See Färe and Primont (1995) or Shephard (1970) for further discussion of this distance function.

To define the directional output distance function, a directional vector $g \in \Re_{+}^{M}$, $g \neq 0$ is required. This vector determines the direction in which the production frontier is approached. In case of Shephard it is the output vector itself, i.e. $g = y$. Thus, the directional distance function is defined as

$$\vec{D}(x,y;g) = \sup\{\psi > 0 : (y + \psi g) \in P(x)\} . \tag{3}$$

This function meets:

iii) Translation: $\vec{D}(x, y + \alpha g; g) = \vec{D}(x,y;g) - \alpha, \quad \alpha \in \Re$,

iv) Representation: $\vec{D}(x,y;g) \ge 0$ *if and only if* $y \in P(x)$.

The translation property follows from the definition of the function and the representation property is due to the assumption of strong disposability of outputs. See Färe and Grosskopf (2004) for more on the properties of directional output distance functions.

Both distance functions meet representation and are therefore dual to the revenue function,

$$R(x,p) = \max\{py : y \in P(x)\}, \tag{4}$$

where $p \in \Re_{+}^{M}$ denotes output prices. If $g = y$ then

$$\vec{D}(x,y;g) = \frac{1}{D(x,y)} - 1. \tag{5}$$

To generate parametric functional forms for the two types of distance functions we will restrict ourselves to the class of functions that is linear in parameters. This class has been referred to as the class of transformed quadratic functions [Diewert (2002)], or the functions that can be represented as second-order Taylor series approximations [Färe and Sung (1986)]. For any $q \in \Re^{I}$ with component q_i, and twice differentiable functions $h : \Re \rightarrow \Re$ and $\rho : \Re \rightarrow \Re$ with an inverse ρ^{-1}, a function in this class can be written as

$$F(q) = \rho^{-1}\left(\alpha_0 + \sum_{i=1}^{I} a_i h(q_i) + \sum_{i=1}^{I}\sum_{j=1}^{J} a_{ij} h(q_i) h(q_j)\right). \tag{6}$$

Färe and Sung (1986)[1] solve for functions that are simultaneously transformed quadratic and homogeneous of degree one, i.e. functions that meet (6) and *i)*, and find two solutions: the translog function [Christensen *et al.* (1971)],

(7)

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[1] The authors assume I=2.

$$F(q) = a_0 + \sum_{i=1}^{I} a_i \ln q_i + \sum_{i=1}^{I} \sum_{j=1}^{J} a_{ij} \ln q_i \ln q_j$$

and the quadratic mean of order *r* function,

(8) $$F(q) = \left(\sum_{i=1}^{I} a_i q_i^r + \sum_{i=1}^{I} \sum_{j=1}^{J} a_{ij} q_i^{r/2} q_j^{r/2}\right)^{1/r}.$$

Färe and Lundberg (2005)[2] solve for functions that are simultaneously transformed quadratic and satisfy the translation property, i.e. functions that meet (6) and *iii)*, and find the only two solutions: the quadratic function,

(9) $$F(q) = a_0 + \sum_{i=1}^{I} a_i q_i + \sum_{i=1}^{I} \sum_{j=1}^{J} a_{ij} q_i q_j$$

and

(10) $$F(q) = \frac{1}{2\lambda} \ln \sum_{i=1}^{I} \sum_{j=1}^{J} a_{ij} \exp(\lambda q_i) \exp(\lambda q_j).$$[3]

The translog and the quadratic functions are the only ones we will compare in this paper. These functions have both first- and second-order terms, while the other two functions have second-order terms only. It is likely that the quality of approximation of the true technology attained by these specifications will be different, thus a reliable

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[2] The authors assume that $\alpha_0 = 0$ in (6).

[3] These two functions were suggested for the parameterization of the directional distance function by Chambers (1998) who took $\lambda = 1/2$ in (10).

benchmark is needed. We investigate the quality of these approximations by designing a Monte Carlo experiment that allows us to shed light on the relative strengths and weaknesses of these two models, as well as to pinpoint the factors that can affect the quality of approximation for each of them.

3. Monte Carlo Experiments

In our simulation study we assume that the true technology is known and consider four representations, each a polynomial of order four. We are mainly interested in the shape of the output set, thus we assume that one input (normalized to equal unity) produces two outputs. The technology is given by

$$P(1) = \{(y_1, y_2) : y_2 = \beta_0 + \beta_1 y_1 + \beta_2 y_1^2 + \beta_3 y_1^3 + \beta_4 y_1^4 \equiv f(y_1)\}. \tag{11}$$

The parameter vector $\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)$ models the varying rate of change in the opportunity cost of one output in terms of the other. It is chosen in the following way:

	Model 1	Model 2	Model 3	Model 4
β_0	11.70	11.10	10.60	10.10
β_1	-0.91	-0.72	-0.54	-0.35
β_2	0.50×10^{-5}	0.50×10^{-4}	0.10×10^{-2}	0.02
β_3	0.10×10^{-4}	0.10×10^{-3}	0.10×10^{-2}	0.02
β_4	-0.45×10^{-3}	-0.12×10^{-2}	-0.24×10^{-2}	-0.41×10^{-2}

The corresponding plots of the output set boundaries across the valid range of y_1 are given in Figure 1.

We generate the quantities y_1 by randomly drawing sample sizes (K) of 50, 100, 500, and 1000 observations from a gamma distribution characterized by the density function $p(y_1) = y_1^{\lambda-1} e^{-y_1/\theta} \left(\Gamma(\lambda)\theta^{\lambda}\right)^{-1}$, where $\Gamma(\cdot)$ is the gamma function, $(\lambda,\theta) \in \Re_+^2$. For each of four true models we consider two different values for (λ,θ):

$$\text{Type-A Models— } y_1^A \sim Gamma(\lambda = 5, \theta = 0.5),$$

$$\text{Type-B Models— } y_1^B \sim Gamma(\lambda = 18, \theta = 0.25).$$

The quantities y_2 are then generated as

$$y_2 = f(y_1) - \upsilon, \tag{12}$$

where υ is an exponentially distributed random noise that captures technical inefficiency with the density function $p(\upsilon) = \exp\{-\upsilon\}$. Note that type-A and type-B models differ by the area of $P(1)$ around which the output quantities are clustered. While in type-A models the quantity of the second output is generally greater than that of the first output for the majority of observations in the sample, type-B models are associated with more balanced quantities of y_2 and y_1.

The translog and the quadratic distance functions are given respectively by

$$D(1,y) = \exp\left\{\gamma_0 + \gamma_1 \ln y_1 + \gamma_2 \ln y_2 + \frac{\gamma_{11}}{2}(\ln y_1)^2 + \frac{\gamma_{22}}{2}(\ln y_2)^2 + \gamma_{12} \ln y_1 \ln y_2\right\}, \tag{13}$$

$$(14) \qquad \vec{D}(1, y) = \delta_0 + \delta_1 y_1 + \delta_2 y_2 + \frac{\delta_{11}}{2} y_1^2 + \frac{\delta_{22}}{2} y_2^2 + \delta_{12} y_1 y_2 .$$

The parameters of these functions are computed using the linear programming procedure of Aigner and Chu (1968).[4] We consider 200 replications for each of the eight model types and both parameterizations.

The empirical analogues of the frontier of the production technology can be recovered using the vectors of parameter estimates $\hat{\gamma}$ and $\hat{\delta}$ and by assuming technical efficiency for every observation in the sample, i.e., $\vec{D}^k(1, y; g) = 0$ and $D^k(1, y) = 1$ for all k=1,…,K. Using the first output y_{1k}, we can solve K quadratic equations and simulate the optimal output quantities $y_{2k}^*(\hat{\gamma})$ and $y_{2k}^*(\hat{\delta})$, which puts every observation on the frontier of the estimated set, thereby allowing us to visually assess the quality of approximation provided by the translog and the quadratic parameterizations.[5]

We choose the following three benchmarks in order to investigate the desirability of the parametric approximations: (1) the average Euclidean distance between the true and simulated quantity of the second output; (2) the average relative shadow price discrepancy; and (3) the mean Euclidean distance between the true and estimated Morishima elasticities of substitution [Morishima (1967)] for the two parameterizations.

Our first benchmark is obtained using the true and simulated quantities of the second output. It is defined as

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[4] Linear programming methodology facilitates a straightforward modeling of all of the functions' properties, many of which can be imposed only by means of inequality constraints. See Färe *et al.* (2005) and Vardanyan and Noh (in press) for a more detailed analysis of the computation procedures.

[5] One can alternatively use y_2 to simulate the optimal quantity of the first output.

(15) $$\overline{\Delta}(\cdot) = K^{-1}\sqrt{\sum_{k=1}^{K}\left[y_{2k}^{*}(\cdot) - \left(y_{2k} + \upsilon_k\right)\right]^2}\ .$$

The second benchmark can be interpreted as the average discrepancy between the true and estimated marginal rate of transformation evaluated at frontier points. From duality theory, the relative shadow price can be defined as [Färe and Primont (1995); Färe and Grosskopf (2004)]

(16) $$\frac{p_1}{p_2} = \frac{\partial D(1,y)/\partial y_1}{\partial D(1,y)/\partial y_2} = \frac{\partial \vec{D}(1,y;g)/\partial y_1}{\partial \vec{D}(1,y;g)/\partial y_2}\,.$$

Hence, the average Euclidean distance between the true and the estimated relative shadow price is equal to

(17) $$\overline{\Omega}(\hat{\gamma}) = K^{-1}\sum_{k=1}^{K}\sqrt{\left[f'(y_{1k}) + \frac{\partial D(1,y_{1k},y_{2k})/\partial y_1}{\partial D(1,y_{1k},y_{2k})/\partial y_2}\right]^2}$$

and

(18) $$\overline{\Omega}(\hat{\delta}) = -K^{-1}\sum_{k=1}^{K}\sqrt{\left[f'(y_{1k}) + \frac{\partial \vec{D}(1,y_{1k},y_{2k};g)/\partial y_1}{\partial \vec{D}(1,y_{1k},y_{2k};g)/\partial y_2}\right]^2}\ ,$$

where $f'(y_{1k})$ is the true shadow price for observation *k*.

Finally, our third benchmark assesses the relative error in the approximation of the Morishima elasticity of substitution, a measure of the curvature. It is defined as $\partial \ln\left(p_1/p_2\right) / \partial \ln\left(y_2/y_1\right)$ and we have

(19) $$e(\hat{\gamma}) = 1 - \frac{\partial^2 \ln D(1,y) / \partial(\ln y_1)^2}{\partial \ln D(1,y) / \partial \ln y_1} + \frac{\partial^2 \ln D(1,y) / \partial \ln y_1 \partial \ln y_2}{\partial \ln D(1,y) / \partial \ln y_2},$$

and

(20) $$e(\hat{\delta}) = y_1 \left(\frac{\partial^2 \vec{D}(1,y,g) / \partial y_1 \partial y_2}{\partial \vec{D}(1,y,g) / \partial y_2} - \frac{\partial^2 \vec{D}(1,y,g) / \partial {y_1}^2}{\partial \vec{D}(1,y,g) / \partial y_1} \right).$$

Therefore, the average Euclidean distance between the true and estimated elasticity is equal to

$$\overline{\mathrm{E}}(\hat{\gamma}) = K^{-1} \sum_{k=1}^{K} \sqrt{\left[y_{1k} \frac{f''(y_{1k})}{f'(y_{1k})} + 1 - \frac{\partial^2 \ln D(1, y_{1k}, y_{2k}) / \partial(\ln y_1)^2}{\partial \ln D(1, y_{1k}, y_{2k}) / \partial \ln y_1} + \frac{\partial^2 \ln D(1, y_{1k}, y_{2k}) / \partial \ln y_1 \partial \ln y_2}{\partial \ln D(1, y_{1k}, y_{2k}) / \partial \ln y_2} \right]^2},$$

(21)

and

$$\overline{\mathrm{E}}(\hat{\delta}) = K^{-1} \sum_{k=1}^{K} \sqrt{\left[y_{1k} \frac{f''(y_{1k})}{f'(y_{1k})} + y_1 \left(\frac{\partial^2 \vec{D}(1, y_{1k}, y_{2k}, g) / \partial y_1 \partial y_2}{\partial \vec{D}(1, y_{1k}, y_{2k}, g) / \partial y_2} - \frac{\partial^2 \vec{D}(1, y_{1k}, y_{2k}, g) / \partial {y_1}^2}{\partial \vec{D}(1, y_{1k}, y_{2k}, g) / \partial y_1} \right) \right]^2}.$$

(22)

4. Results

The results of our experiment are summarized in Table 1. For each of the quadratic models we specify three directional vectors $g \equiv (g_{y_1}, g_{y_2})$, namely *g*=(1,5), *g*=(1,1), and *g*=(5,1).[6] The translog estimates are also assessed using the three benchmarks discussed earlier.

We detect several tendencies from this comparison. First, the quadratic models exhibit much better global behavior than the translog models, as attested by the frontiers in Figure 2 and Figure 3. Compared to the translog specification, the quadratic parameterizations are associated with smaller mean values of all of the benchmarks we have specified. This tendency is most pronounced in Model 1, but gradually decreases in Models 2 through 4. The sole exception is the average shadow price discrepancy, which in small samples is greater in quadratic specifications that assume $g = (5,1)$, than in the translog model. This result is of hardly any consequence in light of a somewhat surprising curvature of the translog frontiers estimates that are clearly in violation of the global regularity conditions.

Second, the global behavior of type-A models differs from that of type-B models for both of our algorithms. Both the average relative shadow price discrepancy ($\overline{\Omega}$) and the mean distance between the true and estimated elasticity of substitution ($\overline{\mathrm{E}}$) are smaller in type-A quadratic models, although no clear tendencies have been detected with regard to the mean Euclidean distance. A similar trend has been confirmed in the translog parameterizations as well.

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[6] Note that the method that we use to add technical inefficiency in (12) ignores different scale effects entailed by these directional vectors. For a detailed discussion of scaling using directional vectors see Färe and Grosskopf (2004).

Third, we have detected no clear tendencies with regard to the changes in the degree of approximation in small versus big samples. Both $\overline{\Delta}$ and $\overline{\Omega}$ decrease steadily in type-1 quadratic parameterizations with an increase in sample size. However, in Model 3 and especially in Model 4, this tendency no longer holds, and the quality of approximation starts to deteriorate in big samples. For example, the proportion of cases in which the value of a benchmark decreases as a consequence of an increase in *K* equals 81% in Model 1, 61% in Model 2, 56% in Model 3, and only 37% in Model 4. In the translog model this fraction falls from 33% in Model 1 to just 11% in Model 4, as its behavior is in fact noticeably worse in big samples than in small ones. Curiously, the translog parameterization is so constraining that it produces larger values of $\overline{\Delta}$, $\overline{\Omega}$, and $\overline{\mathrm{E}}$ as the sample size increases.

Finally, among four different technologies that we consider the best quality of approximation is achieved in Model 1, which is associated with the lowest average marginal rate of transformation. After that, the performance steadily deteriorates in both the translog and quadratic parameterizations.[7] One possible explanation to this trend may be our assumption regarding the true production technologies, which are polynomials of order four, whereas the functions that are used to approximate them are the processes of order two. Consequently, the second-order parameterizations would be better suited for the approximation of frontiers that are relatively "flat," such as in Model 1 or Model 2.

Hence, our results indicate that the quadratic directional distance functions have better approximation properties compared to the translog models. The quadratic parameterizations are dependable and perform well in both small and big samples. On the

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[7] The only exception is Model 4B, which yields a better approximation than Model 3B in the quadratic specification based on $g = (1,5)$.

contrary, the behavior of the translog models is rather disappointing, as they consistently violate the global regularity conditions.

5. Conclusion

The modeling of production technologies with distance functions can be performed using either a translog Shephard distance function or a quadratic directional distance function. In this paper we have compared the performance of these parameterizations by means of a Monte Carlo experiment. Our results indicate that quadratic models consistently outperform translog parameterizations. We also demonstrate that the translog models are characterized by rather poor economic approximation properties and are repeatedly in violation of the global regularity conditions.

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Figure 1
The True Frontiers of the Output Set

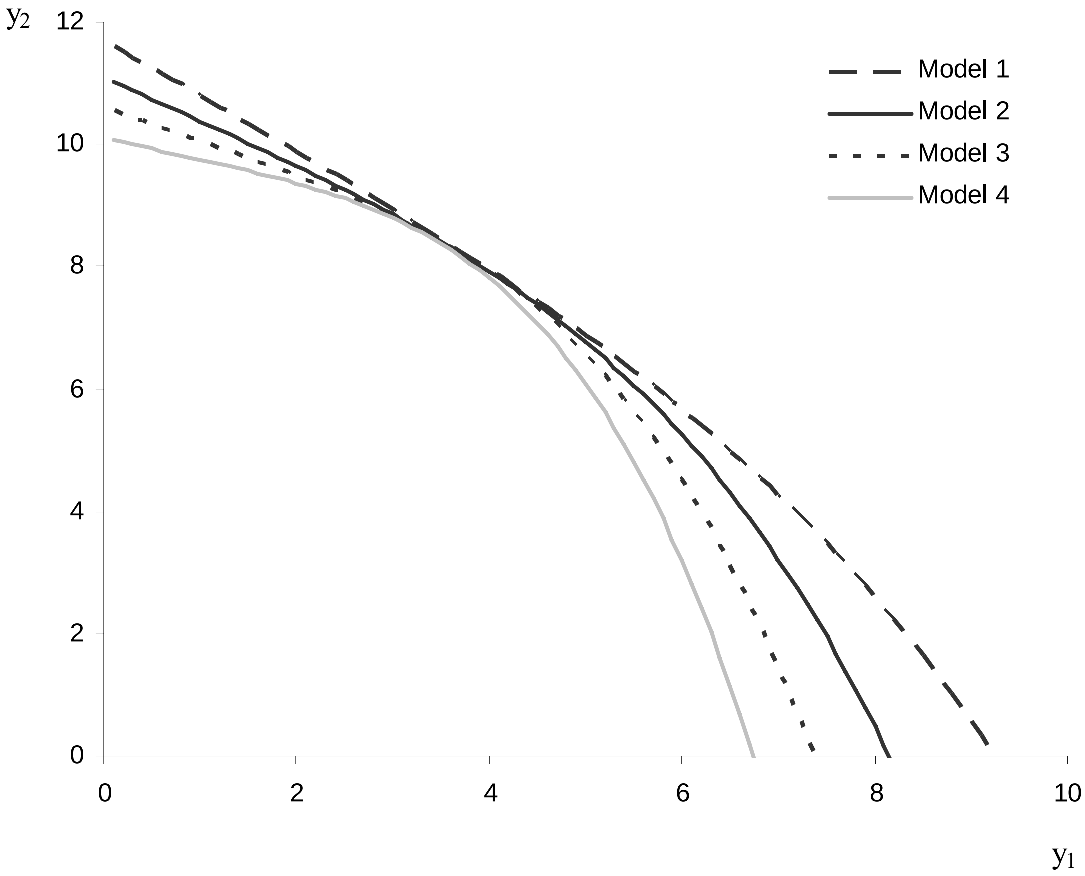

Table 1
Results of the Monte Carlo Experiment

Model 1A	$\overline{\Delta}(\cdot)$	$\overline{\Omega}(\cdot)$	$\overline{\mathrm{E}}(\cdot)$
Quadratic Models			
$g = (1,5)$			
K=50	59.76	67.04	140.41
K=100	31.55	39.78	98.95
K=500	12.22	21.92	77.30
K=1000	10.70	19.59	76.03
$g = (1,1)$			
K=50	67.02	106.79	102.69
K=100	34.42	63.90	83.14
K=500	12.96	35.99	80.19
K=1000	11.26	35.74	82.74
$g = (5,1)$			
K=50	67.56	230.31	101.54
K=100	35.52	104.46	99.73
K=500	15.53	63.83	105.86
K=1000	13.57	56.57	108.43
Translog Model			
K=50	98.84	164.35	647.52
K=100	90.55	158.73	551.93
K=500	125.34	192.68	546.31
K=1000	144.16	216.36	548.79

Model 1B	$\overline{\Delta}(\cdot)$	$\overline{\Omega}(\cdot)$	$\overline{\mathrm{E}}(\cdot)$
Quadratic Models			
$g = (1,5)$			
K=50	67.39	89.37	279.94
K=100	33.75	50.08	186.47
K=500	13.13	31.83	170.87
K=1000	13.00	31.25	176.38
$g = (1,1)$			
K=50	68.34	144.20	213.38
K=100	32.30	84.73	223.58
K=500	11.38	65.29	248.05
K=1000	10.47	63.16	256.66
$g = (5,1)$			
K=50	61.01	224.52	369.74
K=100	37.52	170.01	385.87
K=500	8.48	113.66	402.12
K=1000	7.22	111.73	403.53
Translog Model			
K=50	82.40	152.69	398.71
K=100	78.16	147.96	460.36
K=500	124.07	212.74	723.44
K=1000	163.97	238.06	807.04

Note: Each of the mean benchmarks has been multiplied by 10^3.

Table 1 (Continued)

Model 2A	$\overline{\Delta}(\cdot)$	$\overline{\Omega}(\cdot)$	$\overline{E}(\cdot)$
Quadratic Models			
$g = (1,5)$			
K=50	64.96	85.55	252.48
K=100	40.34	63.20	212.78
K=500	25.91	48.49	194.02
K=1000	26.39	46.82	195.31
$g = (1,1)$			
K=50	61.67	142.95	189.39
K=100	38.94	109.54	185.40
K=500	22.20	81.48	198.29
K=1000	23.99	79.52	201.17
$g = (5,1)$			
K=50	68.46	304.27	277.04
K=100	34.79	275.70	277.64
K=500	15.11	197.78	285.39
K=1000	15.45	172.29	288.80
Translog Model			
K=50	121.40	198.05	874.70
K=100	120.68	197.07	806.06
K=500	164.39	242.96	787.57
K=1000	207.94	252.71	783.87

Model 2B	$\overline{\Delta}(\cdot)$	$\overline{\Omega}(\cdot)$	$\overline{E}(\cdot)$
Quadratic Models			
$g = (1,5)$			
K=50	67.43	110.79	397.89
K=100	36.78	78.06	350.22
K=500	24.01	68.91	360.25
K=1000	26.48	69.49	372.15
$g = (1,1)$			
K=50	64.32	231.97	546.39
K=100	31.64	177.96	591.63
K=500	12.96	152.37	619.65
K=1000	13.04	148.69	627.71
$g = (5,1)$			
K=50	72.34	466.22	941.16
K=100	51.32	355.62	955.49
K=500	45.50	275.36	974.69
K=1000	52.77	260.35	983.71
Translog Model			
K=50	167.33	289.24	932.81
K=100	205.07	347.16	1,120.33
K=500	317.37	461.05	1,484.48
K=1000	355.21	501.17	1,590.52

Table 1 (Continued)

Model 3A	$\overline{\Delta}(\cdot)$	$\overline{\Omega}(\cdot)$	$\overline{\mathrm{E}}(\cdot)$
Quadratic Models			
g = (1,5)			
K=50	75.02	116.87	453.33
K=100	50.52	96.34	391.29
K=500	43.73	83.39	360.42
K=1000	47.57	82.98	358.80
g = (1,1)			
K=50	67.39	226.23	331.99
K=100	35.28	181.10	327.03
K=500	23.11	156.25	342.00
K=1000	25.41	147.65	349.63
g = (5,1)			
K=50	98.64	332.87	507.23
K=100	80.90	319.96	534.40
K=500	108.10	299.86	582.96
K=1000	120.81	290.45	592.13
Translog Model			
K=50	170.90	263.43	1,201.77
K=100	191.12	282.66	1,126.75
K=500	216.37	308.25	1,117.12
K=1000	224.57	321.78	1,120.93

Model 3B	$\overline{\Delta}(\cdot)$	$\overline{\Omega}(\cdot)$	$\overline{\mathrm{E}}(\cdot)$
Quadratic Models			
g = (1,5)			
K=50	66.40	123.55	481.46
K=100	37.90	108.05	483.54
K=500	25.59	100.90	531.39
K=1000	30.43	103.76	549.64
g = (1,1)			
K=50	75.11	313.71	997.19
K=100	58.77	299.27	1,087.12
K=500	61.49	275.72	1,155.32
K=1000	66.47	271.01	1,172.54
g = (5,1)			
K=50	112.71	516.08	1,502.53
K=100	107.18	483.92	1,548.79
K=500	123.17	452.09	1,596.20
K=1000	131.66	440.97	1,604.71
Translog Model			
K=50	379.89	664.50	1,790.19
K=100	419.02	665.97	1,943.87
K=500	615.85	850.84	2,283.01
K=1000	641.62	884.40	2,336.93

Table 1 (Continued)

Model 4A	$\overline{\Delta}(\cdot)$	$\overline{\Omega}(\cdot)$	$\overline{E}(\cdot)$
Quadratic Models			
$g = (1,5)$			
K=50	77.90	119.49	713.63
K=100	58.66	113.75	662.14
K=500	63.54	118.86	609.35
K=1000	70.51	122.31	600.01
$g = (1,1)$			
K=50	78.60	280.44	525.77
K=100	52.15	261.52	560.38
K=500	61.30	244.91	642.95
K=1000	71.53	246.18	663.31
$g = (5,1)$			
K=50	212.82	552.48	939.90
K=100	232.60	564.23	991.52
K=500	304.80	517.25	1,053.56
K=1000	318.07	496.94	1,062.38
Translog Model			
K=50	243.34	370.43	1,635.31
K=100	246.32	375.10	1,602.34
K=500	291.89	421.34	1,556.41
K=1000	306.58	438.19	1,580.70

Model 4B	$\overline{\Delta}(\cdot)$	$\overline{\Omega}(\cdot)$	$\overline{E}(\cdot)$
Quadratic Models			
$g = (1,5)$			
K=50	70.15	166.12	621.39
K=100	33.86	147.16	651.55
K=500	14.29	143.51	693.14
K=1000	15.57	144.49	709.00
$g = (1,1)$			
K=50	122.96	484.57	1,540.96
K=100	102.58	452.98	1,620.32
K=500	116.62	436.95	1,708.13
K=1000	124.15	427.80	1,724.34
$g = (5,1)$			
K=50	162.83	707.34	2,059.59
K=100	170.04	718.70	2,124.91
K=500	188.62	658.35	2,179.76
K=1000	198.69	638.71	2,189.48
Translog Model			
K=50	629.23	1,004.05	2,486.82
K=100	790.47	1,182.93	2,729.05
K=500	991.90	1,366.33	2,958.03
K=1000	1,026.88	1,396.99	2,999.80

Figure 2

The True and Estimated Frontiers of the Output Set; Model 1A, *K*=50

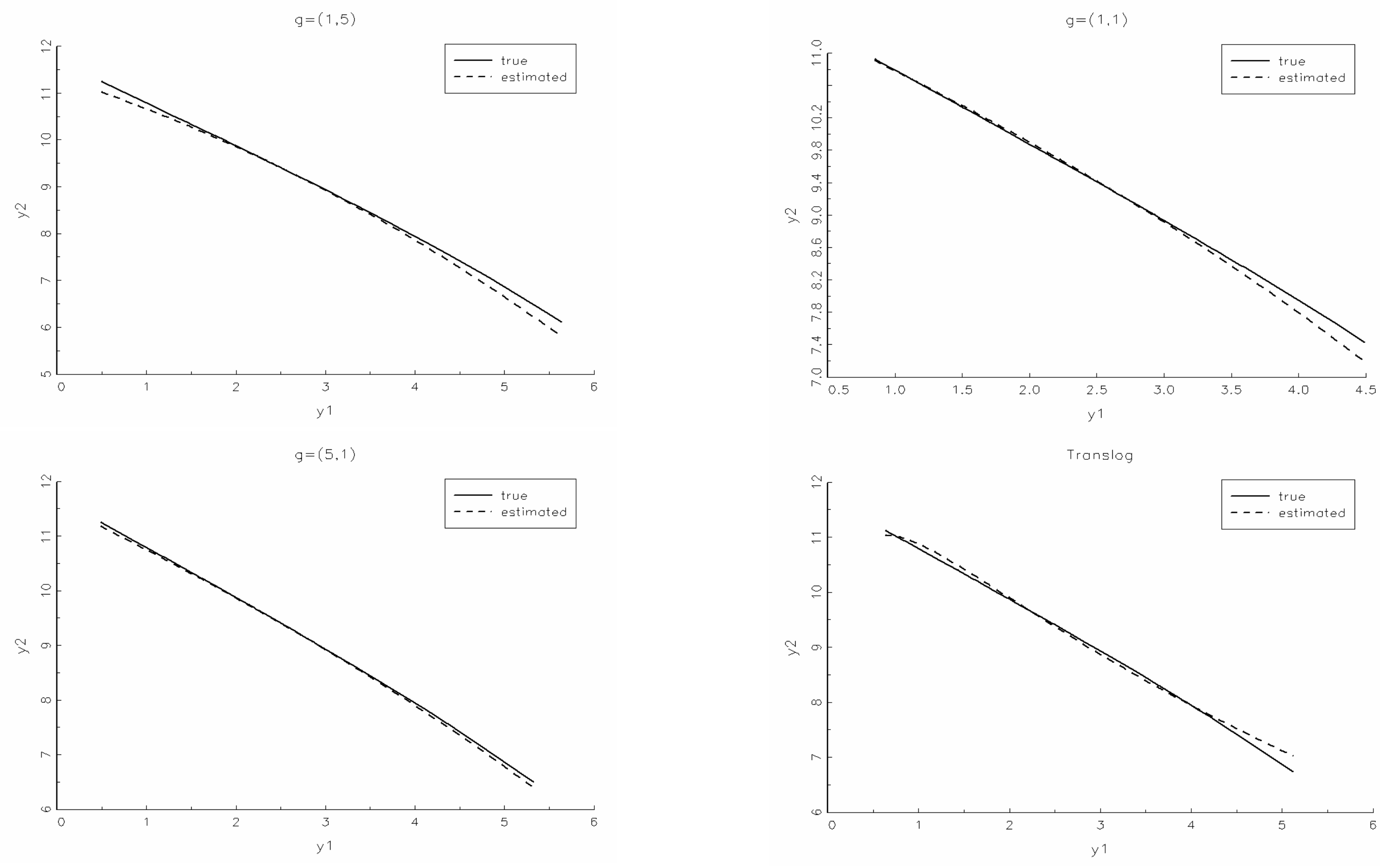

Figure 2 (Continued)
Model 1A, *K*=1000

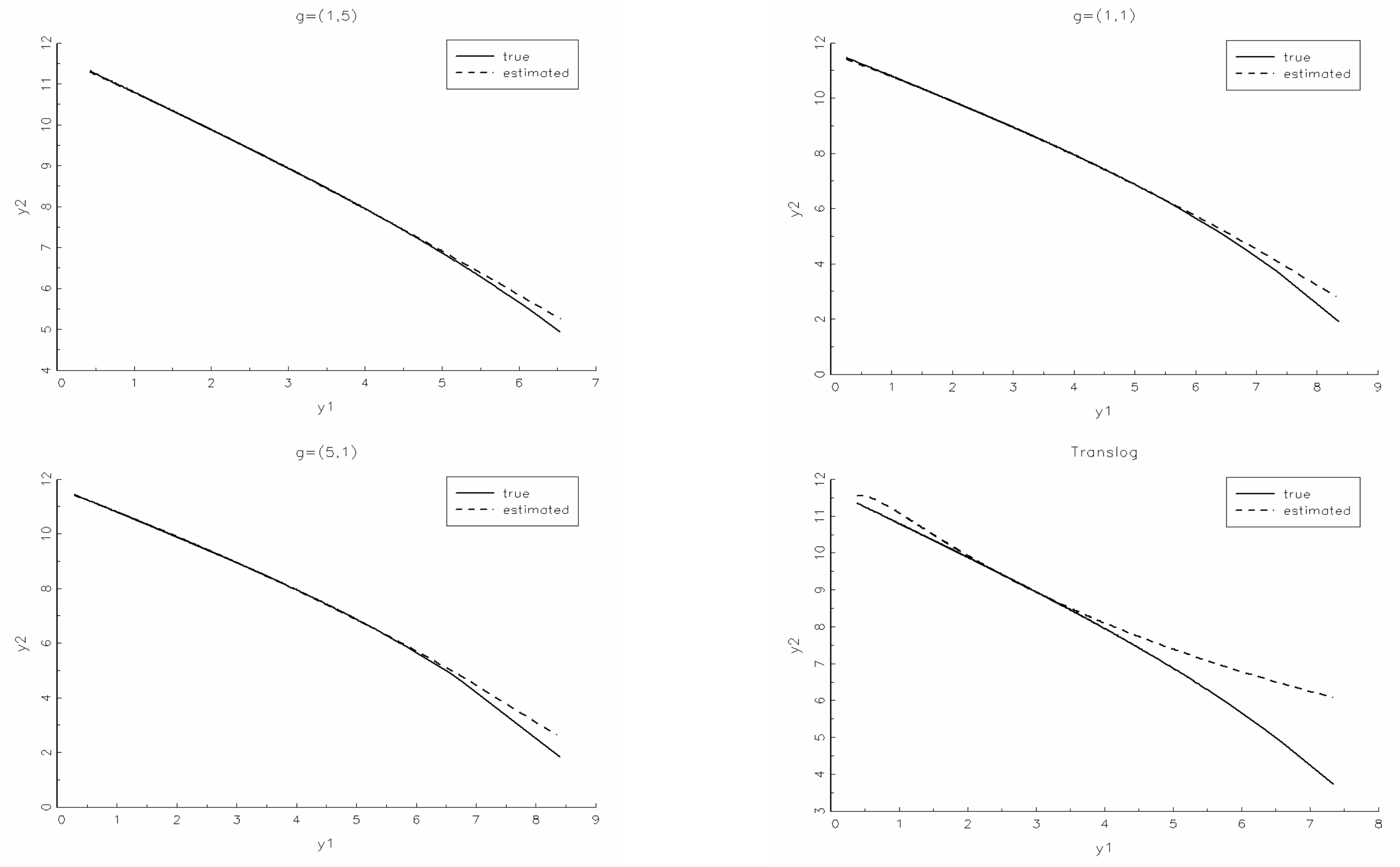

Figure 3
The True and Estimated Frontiers of the Output Set; Model 4B, *K*=50

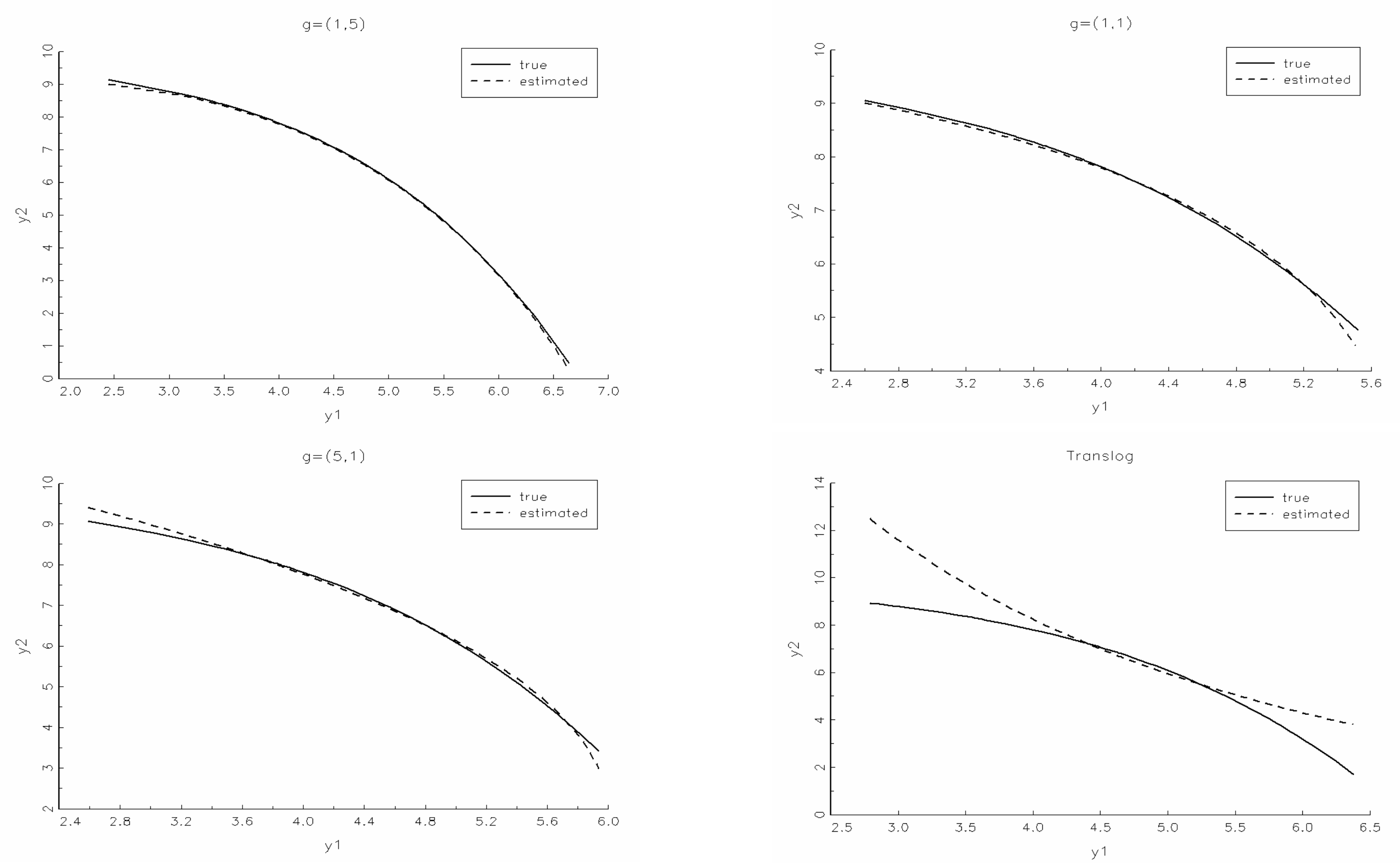

Figure 3 (Continued)
Model 4B, *K*=1000