Universal Covering Space

**Def:** Universal Covering Space

**Lemma:** Let $p : (E, e_0) \to (B, b_0)$ be a covering map, where $B$ is *path connected and locally path connected*, but $E$ need not be path connected. If $E_0$ is a path component of $E$ then $p_0 = p|_{E_0}$ is a covering map.

**Examples:**

Compositions of Covering Maps

**Lemma:** Suppose $p = r \circ q$.

- If $p$ and $r$ are covering maps, $q$ is.
- If $p$ and $q$ are covering maps, $r$ is.
- If $Z$ has a universal covering space, and $r$ and $q$ are covering maps, so is $p$.

Universal Cover

**Theorem:** Let $p : (E, e_0) \to (B, b_0)$ be a universal covering map and $r : (Y, y_0) \to (B, b_0)$ be an covering map. Then there is a covering map $q : E \to Y$ such that $p = r \circ q$.

**Lemma:** Let $p : (E, e_0) \to (B, b_0)$ be a universal covering map. Then $b_0$ has a neighborhood $U$ such that inclusion $i : U \to B$ induces the trivial homomorphism on $\pi_1$.

**Def:** Any space satisfying the condition in the previous Lemma is said to be *semilocally simply connected*.

**Examples:**

Existence of Covers

**Theorem:** (Existence of Covering Spaces:) Let $B$ be path connected, locally path connected, semilocally simply connected, and $b \in B$. Then, given a subgroup $H$ of $\pi_1(B, b)$, there exists a covering map $p : (E, e) \to (B, b)$ with $p_*(\pi_1(E, e)) = H$.

**Corollary:** $B$ has a universal covering space if and only if $B$ is path connected, locally path connected, and semilocally simply connected.