Read Sec. 20, 21,
(Metric Topology)

Def. A sequence (x_n) in X converges to p if for each neighborhood U of p, there exists an N so that if $n > N$, then $x_n \in U$.

Theorem: If X is Hausdorff, $(x_n) \to p$ and $(x_n) \to q$, then $p = q$.

Theorem: A countable product of metric spaces, with the product topology, is metrizable.

Def. A **neighborhood base** at a point x in X is a collection of neighborhoods of x, $\{U_\alpha\}$ such that every neighborhood of x contains one of the $\{U_\alpha\}$.

Def. X is **1st countable** if every point in X has a countable neighborhood base. X is **2nd countable** if X has a countable basis.

Note: Metric spaces are first countable.

Sequence Lemma: If a sequence of points (a_n) in a subset A of X converges to p in X, then $p \in \overline{A}$. Conversely, if X is first countable and $p \in \overline{A}$, then there is a sequence of points in A converging to p.

Theorem: An uncountable product of nontrivial metric spaces X_α with the product topology is not metrizable.

Proof:
Fix distinct points a_α and b_α in each X_α.
Let $A = \{(x_\alpha) \in \prod X_\alpha | x_\alpha \in \{a_\alpha, b_\alpha\}$ and $x_\alpha = a_\alpha$ for only finitely many $\alpha\}$
Let p be the point with αth coordinate equal to a_α for each α.
Claim, p is in $\overline{A}$, but no sequence in $\overline{A}$ converges to p.

Theorem: $\overline{S_\Omega}$ is not metrizable.

Proof: The sequence lemma fails.

Theorem: R^ω with the box topology is not metrizable.
Proof: The sequence lemma fails.

Theorem: Let $f : X \to Y$ be continuous where X is 1st countable. Then f is continuous if and only if whenever $(a_n) \to a$ in X, $(f(a_n)) \to f(a)$ in Y.
Proof:
$\Rightarrow$ True for any X.

$\Leftarrow$ Show that $f(\overline{A}) \subset \overline{f(A)}$.

Theorem: The sum, difference, product and quotient operations from $R \times R$ to R are continuous where defined. The sum, product, difference and quotient of continuous real valued functions are continuous where defined.

Proof: Advanced Calculus

Uniform Limit Theorem: If Y is metrizable and $f_n : X \to Y$ is a sequence of continuous functions converging **uniformly** to a function $f : X \to Y$, then f is continuous.

Another way of stating this is that the subspace of Y^X consisting of continuous functions is closed in Y^X if Y^X is given the uniform bounded metric:
$d(f,g) = \sup\{\overline{d}(f(x), g(x))\}$.