Limits of Functions
(Part I – two-sided limits)

STANDING CONVENTION: $f : \text{dom}(f) \to \mathbb{R}$ and $\text{dom}(f) \supset I$ which is a nonempty open interval that contains the point $a$.

DEF. The function $f$ has limit $L \in \mathbb{R}$ as $x$ approaches $a$ if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

Notation: If $f$ has limit $L$ as $x$ approaches $a$ we write

$$\lim_{x \to a} f(x) = L$$

or

$$f(x) \to L \text{ as } x \to a$$

DEF. If a function has a finite limit we say it converges to its limit.

Facts:

$$\lim_{x \to a} mx + b = ma + b.$$  
A function may not have a limit as $x \to a$.  
Limits are unique if they exist.
What Happens AT $x = a$ does NOT affect the limit as $x \to a$

Only the behavior of the function near $a$ but not at $a$ determines whether there is a limit at $a$.

The following result is used, often implicitly, in many limit calculations:

**Th.** If

\[
\begin{align*}
\text{dom}(f) & \supset I \setminus \{a\}, \quad \text{dom}(g) \supset I \setminus \{a\}, \\
f(x) &= g(x) \quad \text{for all } x \in I \setminus \{a\} \\
\exists \lim_{x \to a} g(x)
\end{align*}
\]

then

\[
\exists \lim_{x \to a} f(x) = \lim_{x \to a} g(x)
\]
Sequential Characterization of Limits of Functions

The following result enables us to transfer most everything we know about limits of sequences to corresponding results about limits of functions.

**Th. (SCLF)** The following are equivalent:
1. \( \lim_{x \to a} f(x) = L \).
2. For every sequence \( \{x_n\}_{n=1}^{\infty} \) in \( I \setminus \{a\} \) with limit \( a \),
   \( \lim_{n \to \infty} f(x_n) = L \).

**Remarks.**
1. This theorem also is true for the infinite limits \( L = \pm \infty \) and/or for limits as \( x \) approaches \( \pm \infty \). (Of course we haven’t defined these limits yet but you should be able to!)

2. In the theorem when \( a \in \mathbb{R} \), the interval \( I \) can be any open interval containing \( a \). In the case that \( x \to \infty \), \( I \) can be any open interval of the form \((b, \infty)\). What about when \( x \to -\infty \)?
The Algebra of Functions

Given functions \( f : X \to \mathbb{R} \) and \( g : X \to \mathbb{R} \) and \( \alpha \in \mathbb{R} \), we define their sum, difference, scalar multiple, product, and quotient, which are denoted respectively by

\[
\begin{align*}
&f + g, \quad f - g, \quad \alpha f, \quad fg, \quad \frac{f}{g}
\end{align*}
\]

as follows:

1. \( f + g \) is the function whose value at \( x \) is

\[
(f + g)(x) = f(x) + g(x)
\]

2. \( f - g \) is the function whose value at \( x \) is

\[
(f - g)(x) = f(x) - g(x)
\]

3. \( \alpha f \) is the function whose value at \( x \) is

\[
(\alpha f)(x) = \alpha f(x)
\]

4. \( fg \) is the function whose value at \( x \) is

\[
(fg)(x) = f(x)g(x)
\]

5. \( \frac{f}{g} \) is the function whose value at \( x \) is

\[
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}
\]

The domain of each of the functions \( f + g, f - g, \alpha f, fg \) is \( \text{dom}(f) \cap \text{dom}(g) \) and the domain of \( \frac{f}{g} \) is \( \{ x \in \text{dom}(f) \cap \text{dom}(g) : g(x) \neq 0 \} \).
Explicit Calculations With Limits

The process of taking a limit interacts as you would expect with the basic operations $+, -, \times, \text{ and } \div$ and with the order properties of real numbers.

**Th.** Suppose $\exists \lim_{x \to a} f(x)$ and $\exists \lim_{x \to a} g(x)$ and $c \in \mathbb{R}$. Then the limits on the left all exist and have the indicated value:

\[
\begin{align*}
\lim_{x \to a} (f(x) + g(x)) &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \\
\lim_{x \to a} (cf(x)) &= c \lim_{x \to a} f(x) \\
\lim_{x \to a} (f(x)g(x)) &= \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)
\end{align*}
\]

If, in addition, $\lim_{x \to a} g(x) \neq 0$, then

\[
\lim_{x \to a} \left( \frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}
\]

**Th.** Suppose $f(x)$ and $g(x)$ have limits as $x \to a$ and

\[
\begin{align*}
f(x) &< g(x) \text{ for all } x \in I \setminus \{a\} \\
or &&
\begin{align*}
f(x) &\leq g(x) \text{ for all } x \in I \setminus \{a\}
\end{align*}
\]

Then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$. 

The Squeeze Law
(AKA The Three Functions Theorem)

**Th.** Suppose $f(x)$, $g(x)$, and $h(x)$ are real-valued functions with

\[
f(x) \leq g(x) \leq h(x) \quad \text{for all large } x \in I \setminus \{a\}
\]

\[
\exists \lim_{x \to a} f(x) \quad \text{and} \quad \exists \lim_{x \to a} h(x) \quad \text{and} \quad \lim_{x \to a} f(x) = \lim_{x \to a} h(x).
\]

Then

\[
\exists \lim_{x \to a} g(x) \quad \text{and all three limits are equal.}
\]

Corollaries:

1. \[|g(x)| \leq h(x) \to 0 \implies \exists \lim_{x \to a} g(x) = 0.\]

2. If $f(x)$ is bounded near $a$ (which means $\exists M > 0$ such that $|f(x)| \leq M$ for all $x \in I \setminus \{a\}$) and if $g(x) \to 0$ as $x \to a$, then $\exists \lim_{x \to a} f(x) g(x) = 0$. 
One-Sided Limits

DEF. (right-hand limit) If dom \( f \) contains some open interval with left end-point \( a \in \mathbb{R} \) and \( L^+ \in \mathbb{R} \), then

\[
\lim_{x \to a^+} f(x) = L^+
\]

means: Given any \( \varepsilon > 0 \), \( \exists \delta > 0 \) such that

\[
0 < x - a < \delta \implies |f(x) - L^+| < \varepsilon
\]

DEF. (left-hand limit) LTR

Th. Let \( I \) be an open interval, \( a \in I \), and dom \( f \supset I \setminus \{a\} \). Then

\[
\exists \lim_{x \to a} f(x) \iff \exists \lim_{x \to a^-} f(x), \ \exists \lim_{x \to a^+} f(x) \text{ and } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)
\]

in which case all three limits are equal.

Notation:

\[
\lim_{x \to a^+} f(x) = L^+ = f(a+) \text{ and } \lim_{x \to a^-} f(x) = L^- = f(a-)
\]

Remark: All the algebraic limit laws and squeeze laws hold when two-sided limits are replaced by one-sided limits.
Limits Involving Infinity

DEF. (finite limit as \( x \to \infty \)) If \( \text{dom}(f) \) contains an open interval of the form \((c, \infty)\) for some \( c \) and \( L \in \mathbb{R} \), then

\[
\lim_{x \to \infty} f(x) = L
\]

means: Given any \( \varepsilon > 0 \) \( \exists M > 0 \) such that

\[
x > M \implies |f(x) - L| < \varepsilon
\]

DEF. (finite limit as \( x \to -\infty \)) LTR

Remark: All the algebraic limit laws and squeeze laws hold for finite limits at \( \pm \infty \).

DEF. (limit \( \infty \) as \( x \to a \in \mathbb{R} \)) If \( \text{dom}(f) \supset I \setminus \{a\} \), then

\[
\lim_{x \to a} f(x) = \infty
\]

means: Given any \( M \in \mathbb{R} \) \( \exists \delta > 0 \) such that

\[
0 < |x - a| < \delta \implies f(x) > M
\]

Remark. WLOG you can assume in applying this definition that \( M > M_0 \) for any convenient \( M_0 \). (Proof? CDP.)

It is left to you to give precise definitions for the limits

\[
\lim_{x \to a} f(x) = -\infty, \quad \lim_{x \to a^\pm} f(x) = \pm \infty, \quad \lim_{x \to \pm \infty} f(x) = \pm \infty
\]

using the foregoing definitions as models. (CDP)
Continuous Functions

DEF. Let $f : E \to \mathbb{R}$ and $a \in E$. The function $f$ is **continuous at** $a$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$x \in E \text{ and } |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$ 

If $S \subseteq E$ and $f$ is continuous at each point in $S$, then we say $f$ is **continuous on** $S$. If $f$ is continuous at each point in its domain we say $f$ is **continuous**.

Facts:
1. $f(x) = b$ is continuous for any fixed real number $b$.
2. $f(x) = x$ is continuous.
3. $f(x) = |x|$ is continuous.

Continuity can be characterized sequentially:

**Th. (SC of C)** Let $f : E \to \mathbb{R}$ and $a \in E$. The following are equivalent:
1. $f$ is continuous at $a$.
2. For each sequence $\{x_n\}$ in $E$ with limit $a$, the sequence $\{f(x_n)\}$ has limit $f(a)$.

**Cor.** $f$ is not continuous at $a$, if $\exists$ a sequence $\{x_n\}$ in $E$ with limit $a$, such that $f(x_n)$ does not have limit $f(a)$.

**Fact:** The function $f(x) = \sqrt{x}$ is continuous.
The sequential characterization of continuity and the algebraic limit laws for sequences yield:

Th. Let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be continuous at $a \in E$ and let $\alpha \in \mathbb{R}$. Then the functions $f + g$, $f - g$, $\alpha f$, and $fg$, are continuous at $a$. Moreover, if $g(a) \neq 0$, the function $f/g$ is continuous at $a$.

Cor. All polynomial functions and rational functions are continuous.

Very often continuity can be expressed conveniently in terms of limits of functions:

Th. Let $f : E \to \mathbb{R}$ and $a \in E$. If $a$ belongs to an open interval $J$ and $J \subset E$, then the following are equivalent:
1. $f$ is continuous at $a$.
2. $\lim_{x \to a} f(x) = f(a)$.

Th. Let $I$ be an interval (of any type) with endpoints $\alpha < \beta$, $f : I \to \mathbb{R}$. The following are equivalent:
1. $f$ is continuous on $I$.
2. $\lim_{x \to a} f(x) = f(a)$ at each point $a \in I$ with one-sided limits understood at the endpoints of $I$ that belong to $I$. 
Continuity of Composite Functions

DEF. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are functions, the function $g \circ f : X \rightarrow Z$ defined by

$$(g \circ f) (x) = g (f (x))$$

for each $x \in X$ and read $g$ composed with $f$ is called a composite function.

Facts:
1. Even if $g \circ f$ and $f \circ g$ are both defined it is usually the case that $g \circ f \neq f \circ g$.

Th. Let $E$, $F$, and $G$ be subsets of $\mathbb{R}$. Let $f : E \rightarrow F$, $g : F \rightarrow G$, and $a \in E$. Then:
1. If $f$ is continuous at $a$ and $g$ is continuous at $b = f (a)$, then $g \circ f$ is continuous at $a$.
2. If $f$ is continuous and $g$ is continuous, then $g \circ f$ is continuous.
More About Sups and Infs

DEF. Let $E$ be a nonempty subset of real numbers.
1. If $E$ is not bounded above, by definition, $\sup E = \infty$.
2. If $E$ is not bounded below, by definition, $\inf E = -\infty$.

As a consequence of these conventions and the CA for $\mathbb{R}$:

Every nonempty set $E \subset \mathbb{R}$ has a supremum and an infimum.

Facts:
1. For $E \neq \emptyset$, $E$ is bounded above $\iff \sup E < \infty$.
2. For $E \neq \emptyset$, $\exists \{x_n\}$ in $E$ with $x_n \to \sup E$. If $\sup E \notin E$, the sequence can be chosen strictly increasing.
3. What is the situation for $\inf E$?
DEF. A function $f: E \to \mathbb{R}$ is
- **bounded above** if $\exists M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in E$;
- **bounded below** if $\exists m \in \mathbb{R}$ such that $f(x) \geq m$ for all $x \in E$;
- **bounded** if it is bounded above and bounded below. So, $f$ is bounded $\iff \exists M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in E$.

Equivalently, a function is bounded (above, below) if its range, $\text{ran}(f)$ is bounded (above, below).

DEF. A function $f: E \to \mathbb{R}$ is
- **increasing** if $x_1, x_2 \in E$ and $x_1 < x_2 \implies f(x_1) \leq f(x_2)$;
- **decreasing** if $x_1, x_2 \in E$ and $x_1 > x_2 \implies f(x_1) \geq f(x_2)$;
- **strictly increasing** if $x_1, x_2 \in E$ and $x_1 < x_2 \implies f(x_1) < f(x_2)$;
- **strictly decreasing** if $x_1, x_2 \in E$ and $x_1 > x_2 \implies f(x_1) > f(x_2)$;
- **monotone** if it is either increasing or decreasing.
Th. *(Sign Preserving Property)* Let $f : E \to \mathbb{R}$ be continuous at $a \in E$. If $f(a) > 0$, then there is a $\delta > 0$ such that

$$x \in E \text{ and } |x - a| < \delta \implies f(x) > 0.$$ 

Th. *(Extreme or Max-Min Value Theorem)* Let $I$ be a closed, bounded interval and $f : I \to \mathbb{R}$ be continuous. Then $f$ is bounded and $f$ assumes its maximum and minimum values on $I$. That is, there are points $\alpha$ and $\beta$ in $I$ such that

$$f(\alpha) \leq f(x) \leq f(\beta) \text{ for all } x \in I.$$ 

In other words, the range of $f$ is a bounded set and contains its supremum and infimum.
**Th A. (Intermediate Value Theorem)** Let $I = [a, b]$ be a closed, bounded interval and $f : I \rightarrow \mathbb{R}$ be continuous. If $y$ is any value strictly between $f(a)$ and $f(b)$, then there is a point $x \in (a, b)$ such that $f(x) = y$.

Here are two equivalent formulations of the IVT:

**Th B. (Intermediate Value Theorem)** Let $I = [a, b]$ be a closed, bounded interval and $f : I \rightarrow \mathbb{R}$ be continuous such that $f(a)f(b) < 0$. Then there is a point $x \in (a, b)$ such that $f(x) = 0$.

**Th C. (Intermediate Value Theorem)** Let $I$ be an interval of any type (open, closed, half-open, bounded or unbounded) and $f : I \rightarrow \mathbb{R}$ be continuous. Then the range of $f$ is an interval.

**Th. (nth roots exist)** Let $n \in \mathbb{N}$. The function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(x) = x^n$ is continuous, strictly increasing and onto; hence, invertible.

Cor. For each $n \in \mathbb{N}$ and real number $y \geq 0$ there is a unique nonnegative real number $x$ such that $x^n = y$.

Notation: If $y > 0$, the unique $x$ above is the **positive $n$th root of $y$**, denoted by $\sqrt[n]{y}$.
The Exponential Function on $\mathbb{Q}$

Let $a > 0$. By definition:

Step 1. $a^0 = 1$.

Step 2. For $n \in \mathbb{N}$,

$$a^n = \underbrace{a \cdot a \cdots a}_{n-\text{factors}}, \quad a^{-n} = \frac{1}{a^n},$$

$$a^{1/n} = \sqrt[n]{a}$$

Step 3. For $q \in \mathbb{Q}$, write $q = m/n$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then

$$a^q = (a^{1/n})^m$$

Facts:

1. The definition in Step 3 makes sense. That is, if $q = m/n = m'/n'$ with $m, m' \in \mathbb{Z}$ and $n, n' \in \mathbb{N}$, then

$$\left(a^{1/n}\right)^m = \left(a^{1/n'}\right)^{m'}$$

2. The usual rules of exponents hold:

$$a^{q+q'} = a^q a^{q'}, \quad a^{-q} = 1/a^q$$

$$(a^q)^{q'} = a^{qq'}, \quad (ab)^q = a^q b^q \quad (b > 0)$$

3. The function $f : \mathbb{Q} \to \mathbb{R}$ defined by $f(q) = a^q$ is positive, continuous, and strictly increasing if $a > 1$ and is strictly decreasing if $0 < a < 1$. 

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Rational Power Functions

1. For each fixed value of \( x > 0 \) (think of \( x = a \)) and each \( q \in \mathbb{Q} \), the number \( x^q \) has already been defined. (It is just a matter of your point of view.) So we now know what the function

\[
f : [0, \infty) \to \mathbb{R} \quad \text{given by} \quad f(x) = x^q
\]

means for any rational power \( q \).

2. If \( q = m/n \) with \( m \in \mathbb{Z} \) and \( n \) an odd number (including \( n = 1 \)), then (proof?) each \( x \in (-\infty, \infty) \) has a unique \( n \)th root \( x^{1/n} \) and by definition \( x^q = (x^{1/n})^m \). For such \( q \), \( f(x) = x^q \) has domain \((-\infty, \infty)\).

3. The function \( f(x) = x^q \) is continuous on its domain.
Uniform Continuity

DEF. Let $E \subset \mathbb{R}$. A function $f : E \to \mathbb{R}$ is uniformly continuous on $E$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$x, x' \in E \text{ and } |x - x'| < \delta \implies |f(x) - f(x')| < \varepsilon.$$ 

Facts:

1. $f$ uniformly continuous on $E$ implies $f$ is continuous on $E$.
2. Continuity and uniform continuity on a set are not equivalent concepts.

Th. Let $f : E \to \mathbb{R}$ be uniformly continuous. Then $f$ maps Cauchy sequences in $E$ onto Cauchy sequences in $\mathbb{R}$. 
The Exponential Function on $\mathbb{R}$

1. Fix $a > 0$. $f(q) = a^q$ is uniformly continuous on $(\alpha, \beta) \cap \mathbb{Q}$ for any bounded interval $(\alpha, \beta) \subset \mathbb{R}$.

2. If $x \in \mathbb{R}$ and $\{q_n\}$ is any sequence in $\mathbb{Q}$ with limit $x$, then
   \[ \lim_{n \to \infty} a^{q_n}. \]

3. The limit in 2 depends only upon $x$ and not upon the particular sequence $\{q_n\}$ with limit $x$.

The foregoing facts justify the following definition.

DEF. Fix $a > 0$. Let $x \in \mathbb{R}$. By definition
\[ a^x = \lim_{n \to \infty} a^{q_n} \]
where $\{q_n\}$ is any sequence in $\mathbb{Q}$ with limit $x$.

In other words, $a^x$ is the unique continuous extension of $a^q$ from $\mathbb{Q}$ to $\mathbb{R}$.

Th. Fix $a > 0$ with $a \neq 1$. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a^x$ is positive, continuous, has range $(0, \infty)$, satisfies the usual rules of exponents, and is strictly increasing when $a > 1$ and strictly decreasing when $0 < a < 1$. 
DEF. Let $E \subset \mathbb{R}$. The **closure** of $E$, denoted $\overline{E}$, consists of all points $x \in \mathbb{R}$ that are limits of sequences in $E$. That is, $\exists$ a sequence $\{x_n\}$ in $E$ with limit $x$.

DEF. Let $E \subset \mathbb{R}$. $E$ is **closed** if $E = \overline{E}$.

Facts:

1. $E = (0, 1)$ has closure $\overline{E} = [0, 1]$.
2. $\mathbb{Q} = \mathbb{R}$ (another version of $\mathbb{Q}$ is dense in $\mathbb{R}$).
3. $E = (a, b) \cap \mathbb{Q}$ has closure $\overline{E} = [a, b]$.
4. $E = [a, b]$ has closure $\overline{E} = [a, b]$; so $[a, b]$ is closed.

**Th. A** If $E \subset \mathbb{R}$ is closed and bounded and $f : E \to \mathbb{R}$ is continuous, then $f$ is uniformly continuous on $E$.

**Th. B** Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$ be uniformly continuous on $E$. Then $f$ has a unique extension by continuity to $\overline{E}$. That is, there is a unique continuous function $g : \overline{E} \to \mathbb{R}$ (which, in fact, is uniformly continuous on $\overline{E}$) such that $g(x) = f(x)$ for all $x \in E$. 